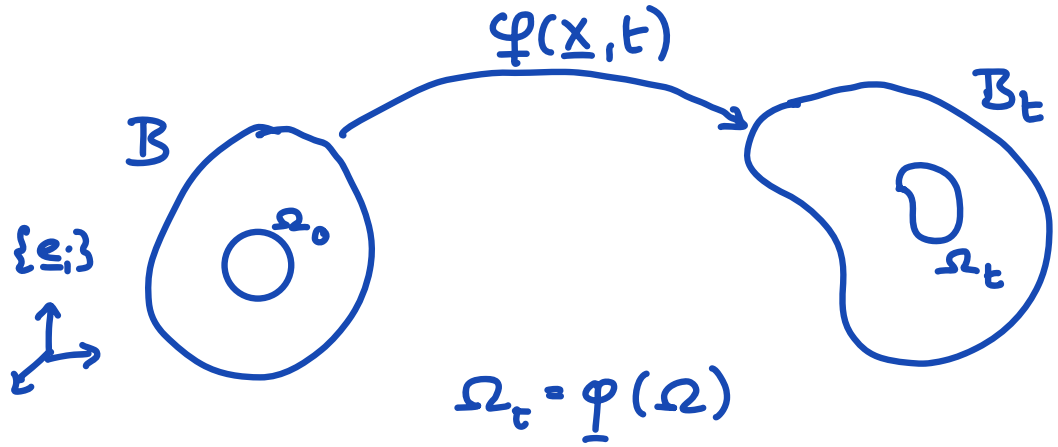
Local Eulerian Balance Laws



Arbitrary subset Ω

I) Conservation of mass

Mass of Ω_t : $M[\Omega_t] = \int_{\Omega_t} \rho(x,t)\, dV_x$

no reactions or relativistic effects $\Rightarrow$ mass is constant

$$\boxed{\frac{d}{dt} M[\Omega_t] = 0} \quad \Rightarrow M[\Omega_t] = M[\Omega_0]$$

Initial density: $M[\Omega_0] = \int_{\Omega_0} \rho_0(X)\, dV_X$

Current density: $M[\Omega_t] = \int_{\Omega_t} \rho(x,t)\, dV_x$

using $dV_x = J(\underline{X},t)\, dV_X$ where $J = \det(\underline{\underline{F}})$

$$M[\Omega_t] = \int_{\Omega_t} \rho(\underline{x},t)\, dV_x = \int_{\Omega_0} \underbrace{\rho(\underline{\varphi}(\underline{X},t),t)}_{\rho_m(\underline{X},t)}\, J(\underline{X},t)\, dV_X$$

$$= \int_{\Omega_0} \rho_m(\underline{X},t)\, J(\underline{X},t)\, dV_X$$

$$\Rightarrow M[\Omega_t] = M[\Omega_0] \Rightarrow \int_{\Omega_0} \rho_m(\underline{X},t)\, J(\underline{X},t)\, dV_X = \int_{\Omega_0} \rho_0(\underline{X})\, dV_X$$

$$\int_{\Omega_0} \left[\rho_m(\underline{X},t)\, J(\underline{X},t) - \rho_0(\underline{X})\right] dV_X = 0$$

by arbitrariness of Ω we have

$$\boxed{\rho_m(\underline{X},t)\, J(\underline{X},t) = \rho_0(\underline{X})}$$

Lagrangian statement of mass conservation. $(\underline{X})$

Convert to Eulerian: $\frac{\partial}{\partial t}$

$$\underbrace{\frac{\partial}{\partial t}\rho_m(\underline{X},t)}_{\dot{\rho}(\underline{x},t)}\, J(\underline{X},t) + \rho_m(\underline{X},t)\, \underbrace{\frac{\partial}{\partial t} J(\underline{X},t)}_{J(\nabla_x \cdot \underline{v})_m} = \frac{\partial}{\partial t}\rho_0(\underline{X}) = 0$$

dividing by J and switching to spatial description

$$\dot{\rho}(\underline{x},t) + \rho(\underline{x},t)\, \nabla_x \cdot \underline{v}(\underline{x},t) = 0$$

$$\Rightarrow \boxed{\dot{\rho} + \rho\, \nabla_x \cdot \underline{v} = 0}$$ local Eulerian form

expanding the material derivative we have

$$\frac{\partial}{\partial t}\rho + \underbrace{\nabla_x \rho \cdot \underline{v} + \rho \nabla_x \cdot \underline{v}}_{\nabla_x \cdot (\rho \underline{v})} = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla_x \cdot (\rho \underline{v}) = 0}$$ conservative local Eulerian form

Time derivative of integrals relative to mass

$$\boxed{\frac{d}{dt}\int_{\Omega_t} \phi(\underline{x},t)\rho(\underline{x},t)\, dV_x = \int_{\Omega_t} \dot{\phi}(\underline{x},t)\rho(\underline{x},t)\, dV_x}$$

where $\phi(\underline{x},t)$ is any spatial scalar, vector or tensor field.

$$\int_{\Omega_t} \phi(\underline{x},t)\rho(\underline{x},t)\, dV_x = \int_{\Omega} \phi_m(\underline{X},t)\underbrace{\rho_m(\underline{X},t)\det \underline{\underline{F}}(\underline{X},t)}_{\rho_0(\underline{X})}\, dV_X$$

$$\int_{\Omega_t} \phi(\underline{x},t)\rho(\underline{x},t)\, dV_x = \int_{\Omega} \phi_m(\underline{X},t)\rho_0(\underline{X})\, dV_X$$

Take derivative

$$\frac{d}{dt}\int_{\Omega_t} \phi(\underline{x},t)\rho(\underline{x},t)\, dV_x = \int_{\Omega} \frac{d}{dt}\phi_m(\underline{X},t)\,\rho_0(\underline{X})\, dV_X$$

$$= \int_{\Omega} \dot{\phi}_m(\underline{X},t)\,\rho_0(\underline{X})\, dV_X$$

$$= \int_{\Omega} \dot{\phi}_m(\underline{X},t)\,\rho_m(\underline{X},t)\det \underline{\underline{F}}(\underline{X},t)\, dV_X$$

$$= \int_{\Omega_t} \dot{\phi}(\underline{x},t)\rho(\underline{x},t)\, dV_x \quad \checkmark$$

Laws of inertia (Galileo & Newton)

linear momentum: $\underline{L}[\Omega_t] = \int_{\Omega_t} \rho(\underline{x},t)\,\underline{v}(\underline{x},t)\,dV_x$

angular momentum: $\underline{j}[\Omega_t]_z = \int_{\Omega_t} (\underline{x}-\underline{z}) \times \rho(\underline{x},t)\,\underline{v}(\underline{x},t)\,dV_x$

With respect to a fixed frame of reference, the rate of change of linear and angular momentum of any $\Omega_t \subseteq B_t$ equal the resultant force & torque about the origin

$$\frac{d}{dt}\underline{L}[\Omega] = \int_{\Omega_t} \rho(\underline{x},t)\,\underline{b}(\underline{x},t)\,dV_x + \int_{\partial\Omega_t} \underline{t}(\underline{x},t)\,dA_x$$

$$\frac{d}{dt}\underline{j}[\Omega]_o = \int_{\Omega_t} \underline{x} \times \rho(\underline{x},t)\,\underline{b}(\underline{x},t)\,dV_x + \int_{\partial\Omega_t} \underline{x} \times \underline{t}(\underline{x},t)\,dA_x$$

In Lecture 12 we assumed $\frac{d\underline{L}}{dt} = \underline{0}$, $\frac{d\underline{j}}{dt} = \underline{0}$

$\Rightarrow$ eqbm equations (static)

Today $\Rightarrow$ transient case.

II) Balance of Linear momentum

For an arbitrary $\Omega_t \subseteq B_t$ we have

$$\frac{d}{dt}\int_{\Omega_t} \rho \underline{v}\, dV_x = \int_{\partial\Omega_t} \underline{t}\, dA_x + \int_{\Omega_t} \rho \underline{b}\, dV_x \qquad (\underline{b} = -\underline{g})$$

need to update

where $\rho, \underline{v}, \underline{t}$ and $\underline{b}$ are spatial fields.

Cauchy stress field: $\underline{t} = \underline{\underline{\sigma}}\,\underline{n}$

$$\frac{d}{dt}\int_{\Omega_t} \rho \underline{v}\, dV_x = \int_{\partial\Omega_t} \underline{\underline{\sigma}}\,\underline{n}\, dA_x + \int_{\Omega_t} \rho \underline{b}\, dV_x$$

using tensor divergence theorem

$$\frac{d}{dt}\int_{\Omega_t} \rho \underline{v}\, dV_x = \int_{\Omega_t} \nabla\cdot\underline{\underline{\sigma}} + \rho \underline{b}\, dV_x$$

using derivative relative to mass

$$\int_{\Omega_t} \rho \dot{\underline{v}} - \nabla\cdot\underline{\underline{\sigma}} - \rho\underline{b}\, dV_x = 0$$

by the arbitraryness of Ω_t, we have

$$\boxed{\rho\dot{\underline{v}} - \nabla\cdot\underline{\underline{\sigma}} = \rho\underline{b}}$$ local Eulerian form

Also referred to as Cauchy's first equation of motion.

For $\dot{\underline{v}} = \underline{0}$: $\nabla\cdot\underline{\underline{\sigma}} + \rho\underline{b} = 0$ $\qquad \underline{b} = -\underline{g}$

$$\nabla\cdot\underline{\underline{\sigma}} = \rho\underline{g} \rightarrow \text{Lecture 12}$$

To rewrite this in conservative form consider the following

$$\rho \dot{\underline{v}} = \rho \frac{\partial \underline{v}}{\partial t} + \rho (\nabla \underline{v}) \underline{v} = \frac{\partial}{\partial t}(\rho \underline{v}) - \frac{\partial \rho}{\partial t} \underline{v} + (\nabla \underline{v})(\rho \underline{v})$$

using mass balance $\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \underline{v})$

$$\rho \dot{\underline{v}} = \frac{\partial}{\partial t}(\rho \underline{v}) + \nabla \cdot (\rho \underline{v}) \underline{v} + (\nabla \underline{v})(\rho \underline{v})$$

using $\nabla \cdot (\underline{a} \otimes \underline{b}) = (\nabla \underline{a}) \underline{b} + \underline{a} \nabla \cdot \underline{b}$ (see HW5 Q4)

$$\rho \dot{\underline{v}} = \frac{\partial}{\partial t}(\rho \underline{v}) + \nabla \cdot (\rho \underline{v} \otimes \underline{v})$$

Hence we have <u>conservative</u> local Eulerian form

$$\boxed{\frac{\partial}{\partial t}(\rho \underline{v}) + \nabla \cdot (\rho \underline{v} \otimes \underline{v} - \underline{\underline{\sigma}}) = \rho \underline{b}}$$

conserved quantity: $\rho \underline{v}$ = linear momentum

advective mom. flux: $\rho \underline{v} \otimes \underline{v}$

diffusive mom. flux: $-\underline{\underline{\sigma}}$

III) Balance of angular momentum

For an arbitrary $\Omega_t \subseteq B_t$ we have

$$\frac{d}{dt}\int_{\Omega_t} \underline{x} \times \rho \underline{v}\, dV_x = \int_{\partial\Omega_t} \underline{x} \times \underline{t}\, dA_x + \int_{\Omega_t} \underline{x} \times \rho \underline{b}\, dV_x$$

The left hand side becomes

$$\frac{d}{dt}\int_{\Omega_t} \rho(\underline{x} \times \underline{v})\, dV_x = \int_{\Omega_t} \rho \frac{d}{dt}(\underline{x} \times \underline{v})\, dV_x =$$

$$= \int_{\Omega_t} \rho(\dot{\underline{x}} \times \underline{v} + \underline{x} \times \dot{\underline{v}})\, dV_x \qquad \dot{\underline{x}} = \underline{v}$$

$$\underline{v} \times \underline{v} = \underline{0}$$

$$= \int_{\Omega_t} \rho(\underline{x} \times \dot{\underline{v}})\, dV_x$$

Substituting Cauchy stress field the r.h.s becomes

$$\int_{\Omega_t} \rho(\underline{x} \times \dot{\underline{v}})\, dV_x = \int_{\partial\Omega} \underline{x} \times \underline{\underline{\sigma}}\,\underline{n}\, dA_x + \int_{\Omega_t} \rho(\underline{x} \times \underline{b})\, dV_x$$

$$\int_{\Omega_t} \underline{x} \times (\rho\dot{\underline{v}} - \rho\underline{b})\, dV_x = \int_{\partial\Omega} \underline{x} \times \underline{\underline{\sigma}}\,\underline{n}\, dA_x$$

substitute linear mom. balance $\rho\dot{\underline{v}} - \rho\underline{b} = \nabla_x \cdot \underline{\underline{\sigma}}$

$$\boxed{\int_{\Omega_t} \underline{x} \times \nabla \cdot \underline{\underline{\sigma}}\, dV_x = \int_{\partial\Omega} \underline{x} \times \underline{\underline{\sigma}}\,\underline{n}\, dA_x}$$

This is exactly the statement we had for the static case in Lecture 12 on Mechanical Equilibrium.

$\Rightarrow$ $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$ extends to transient cases.