University of Texas at Austin, Jackson School of Geosciences
GEO 325C/398C, 2026

Continuum Mechanics

Midterm Exam II – November 12, 2026 [35 pts]

(1) [**10 pts**] **The One with Div and Grad.** Use index notation to prove the product rules for the divergence and gradient for the product of a scalar field, $\phi(\mathbf{x})$ and a vector field, $\mathbf{v}(\mathbf{x})$. In both cases, follow the following steps:

I Before doing any manipulations, determine if the result is a scalar, a vector, or a tensor. Provide a short argument (1 pt).

II Define all appropriate quantities with *independent* indices as sum of their components and appropriate bases (1 pt).

III Substitute and manipulate (3 pts).

The definitions of divergence and gradient are

$$\nabla\phi = \phi_{,i}\mathbf{e}_i, \qquad \nabla\cdot\mathbf{v} = v_{i,i},$$
$$\nabla\mathbf{v} = v_{i,j}\,\mathbf{e}_i\otimes\mathbf{e}_j, \qquad \nabla\cdot\mathbf{S} = S_{ij,j}\,\mathbf{e}_i.$$

Show each step in the calculation, feel free to add comments.

(a) Show that: $\nabla\cdot(\phi\mathbf{v}) = \phi\nabla\cdot\mathbf{v} + \mathbf{v}\cdot\nabla\phi$ [**5 pts**].

(b) Show that: $\nabla(\phi\mathbf{v}) = \mathbf{v}\otimes\nabla\phi + \phi\nabla\mathbf{v}$ [**5 pts**].

(2) [**12 pts**] **The one with the Deformation Map** The viscosity of highly viscous fluids, such as magma, is measured using squeeze flow rheometry. Where a sample is compressed between two parallel plates (Fig. 1*a*). In the simplest case the plates are lubricated so that the sample experience no friction along the boundaries and deforms uniformly.
This problem is in cylindrical geometry and if h_t is the current height of the sample and h_0 is the original height, the motion is given by

$$\mathbf{x} = \varphi(\mathbf{X}, t) = \begin{bmatrix}\varphi_r\\ \varphi_\theta\\ \varphi_z\end{bmatrix} = \begin{bmatrix}\sqrt{h_0/h_t}\,R\\ \Theta\\ h_t/h_0\,Z\end{bmatrix},$$

where $\mathbf{X} = [R,\,\Theta,\,Z]^T$ is the initial location and $\mathbf{x} = [r,\,\theta,\,z]^T$ is the current location.

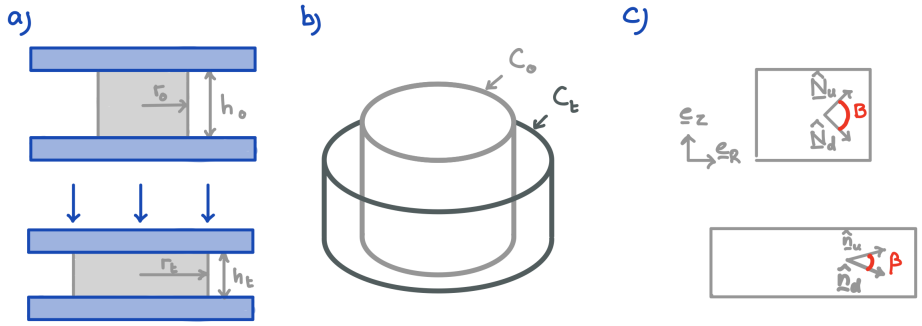


FIGURE 1. (*a*) Cross-section sketch of the squeeze flow rheometry (viscosity measurement). *b* 3D sketch of the change in sample circumference, C_0 to C_t, that gives rise to the hoop strain/stretch. *c* 2D sketch of the two material elements, $\hat{\mathbf{N}}_u$ and $\hat{\mathbf{N}}_d$, in the shear calculation with the initial angle B and the current angle β.

(a) Here we assume that the body is always cylindrical and that it is incompressible. For any cylinder with material radius, $R \leq r_0$, and current radius $r \leq r_t$, use volume conservation ($V_0 = V_t$) to find a relationship between ratio of the radii r/R and the ratio of the heights h_t/h_0. The volume of a cylinder is with radius ρ and height H is $V = \pi\rho^2 H$ **[1 pts]**.

(b) Find the deformation gradient $\mathbf{F} = \nabla\varphi$ and the right Cauchy-Green strain tensor, $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. Keep in mind that the deformation gradient of a vector in cylindrical coordinates is given by

$$\mathbf{F} = \begin{bmatrix} \frac{\partial\varphi_r}{\partial R} & \frac{1}{R}\frac{\partial\varphi_r}{\partial\Theta} & \frac{\partial\varphi_r}{\partial Z} \\ r\frac{\partial\varphi_\theta}{\partial R} & \frac{r}{R}\frac{\partial\varphi_\theta}{\partial\Theta} & r\frac{\partial\varphi_\theta}{\partial Z} \\ \frac{\partial\varphi_z}{\partial R} & \frac{1}{R}\frac{\partial\varphi_z}{\partial\Theta} & \frac{\partial\varphi_z}{\partial Z} \end{bmatrix}.$$

Use the results from (a) to express $\mathbf{F}$ (and hence $\mathbf{C}$) only in terms of ratios between h_t and h_0 **[4 pt]**.

(c) Compute the change in volume during this squeeze flow, $V_x/V_X = J = \det(\mathbf{F})$ to confirm that the material is incompressible **[2 pt]**.

(d) Although the deformation is only in the radial, $\mathbf{e}_R$, and vertical directions, $\mathbf{e}_Z$, there is a stretch in the circumferential direction. From the first Cauchy-Green strain relation we see that, $\lambda(\mathbf{e}_\Theta) = \sqrt{\mathbf{e}_\Theta \cdot \mathbf{C}\,\mathbf{e}_\Theta} = \sqrt{h_0/h_t}$. This is is the so-called hoop-strain that is due to the increase of the circumference as the sample is deformed (Fig. 1b). Use the result from (a) to show that the geometric interpretation of the hoop strain is

$$\lambda(\mathbf{e}_\Theta) = \frac{C_t}{C_0},$$

where $C = 2\pi\rho$ is the circumference of the circle with radius ρ. **[1 pts]**.

(e) This type of flow is often referred to a 'pure elongation' flow which might suggest that there is no shear, γ. Use the second Cauchy-Green strain relation

$$\cos\beta(\hat{\mathbf{N}}_u, \hat{\mathbf{N}}_d) = \frac{\hat{\mathbf{N}}_u \cdot \mathbf{C}\,\hat{\mathbf{N}}_d}{\lambda(\hat{\mathbf{N}}_u)\lambda(\hat{\mathbf{N}}_d)}$$

to show that this deformation induces shear between the two directions $\hat{\mathbf{N}}_u = (\mathbf{e}_R + \mathbf{e}_Z)/\sqrt{2}$ and $\hat{\mathbf{N}}_d = (\mathbf{e}_R - \mathbf{e}_Z)/\sqrt{2}$. Here the shear is the difference between the initial and current angles, $\gamma = B - \beta$, shown in Fig. 1c **[4 pts]**.

(3) **[13 pts] The one with the Material Derivative:** Consider the water gap of the Danube where it cuts through the Franconian Jura range in the Weltenburg narrows (Fig. 2a). The cross-sectional area of the Danube changes dramatically from a deep and narrow pool in the canyon seen in the back of the photograph to a shallow and wide shoal/riffles seen in the front. Due to the dramatic reduction in cross sectional area shown in Fig. 2b the speed of the water increases by an order of magnitude from 0.15 m/s in the pool to 1.5 m/s in the riffles over a distance of only 1750 m. Note that you can see vortices perturbing the water surface in the front while the cliff in the back is called "Stille Wand" (Silent Wall) because the water flow there is so slow it is completely silent.

(a) In this problem we assume the flow is steady and one-dimensional in the downstream direction, so that $\mathbf{v} = [v(x),\, 0,\, 0]^T$. That means we neglect the small velocities in the other directions induced by the change in channel shape. Show that for this 1D case the material derivative for the acceleration reduces to

$$\mathbf{a} = \frac{\partial\mathbf{v}}{\partial t} + (\nabla\mathbf{v})\mathbf{v} = \begin{bmatrix} v\frac{\partial v}{\partial x} \\ 0 \\ 0 \end{bmatrix} \quad \text{or simply} \quad a(x) = v(x)\frac{\partial v(x)}{\partial x}.$$

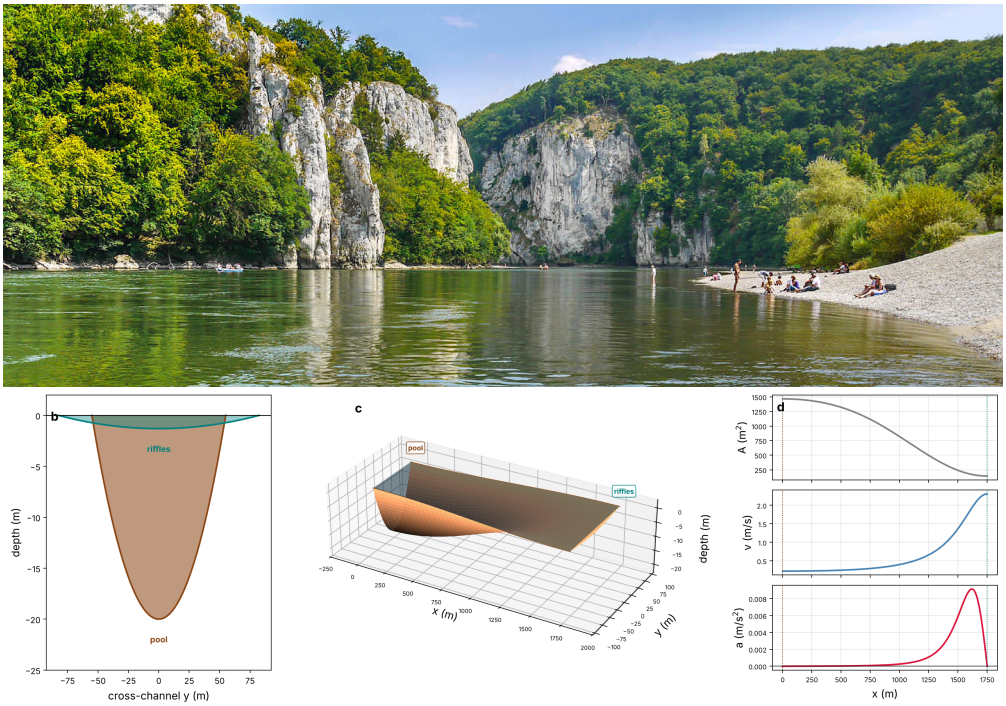


FIGURE 2. (*a*) Photograph of the Weltenburg narrows of the Danube river in Bavaria. People are bathing at the Wipfelsfurt in the front and the deep canyon with the Stille Wand is visible in the back. (*b*) Difference between rectangular and parabolic cross section $A_p = 2/3A_r$. (*c*) Difference in river cross-section at Stille Wand where it is a deep and narrow pool and at Wipfelsfurt where it is a shallow and wide shoal or riffle. (*d*) A 3D rendering of our richer bed geometry for this problem.

Here the velocity gradient is given by $\nabla \mathbf{v} = v_{i,j}\mathbf{e}_i \otimes \mathbf{e}_j$ **[3 pts]**.

(b) We assume that the discharge/flow rate of the river, $Q = A(x)v(x)$, is constant along the segment from the pool to the riffles. Give general expressions for the downstream velocity, $v(x)$, and the downstream acceleration, $a(x)$ in terms of Q, $A(x)$ and $A'(x) = \mathrm{d}A/\mathrm{d}x$. Note, at this point you cannot evaluate the derivative of the area, yet **[3 pts]**.

(c) Now assume the cross-section is parabolic so that $A(x) = \frac{2}{3}w(x)d(x)$, where $w(x)$ is the width and $d(x)$ is the depth of the river (Fig. 2*b*). Both change along the flow direction as simple functions

$$w(x) = w_m - \Delta w \cos(\pi x/L)), \quad \text{and} \quad d(x) = d_m - \Delta d \cos(\pi x/L)$$

where L is the distance between the pool and the riffles, $w_m = (w_r + w_p)/2$ and $d_m = (d_r + d_p)/2$ are the mean width and depth of the river and $\Delta w = (w_r - w_p)/2 > 0$ and $\Delta d = (d_r - d_p)/2 < 0$ are the half of the change in width and depth of the river. Here the subscript p and r refer to the pool and the riffles, respectively. The resulting riverbed geometry is shown in Fig. 2*c*.

Find the derivative of the area, $A'(x) = \mathrm{d}A/\mathrm{d}x$. To keep expressions simple you can give them in terms of $w(x)$ and $d(x)$ as appropriate **[3 pts]**.

(d) Find the expressions for $v(x)$ and $a(x)$ for this particular cross-sectional area by substituting the particular forms of $A(x)$ and $A'(x)$. Again, you can give expression in terms of $w(x)$ and $d(x)$ as appropriate. Fig. 2*d* shows plots of the variation of $A(x)$, $v(x)$, and $a(x)$ **[2 pts]**.

(e) Compute the flow speeds at the pool ($x = 0$) and the riffles ($x = L$) using the following values for the geometry. The width and depth at the pool are $w_p = 110$ m and $d_p = 20$ m and the with and the depth of the riffles are $w_r = 165$ m and $d_r = 1.3$ m (Fig 2*b*). The mean annual discharge measured at a nearby hydrological station is $Q = 330$ m^3/s **[2 pts]**.