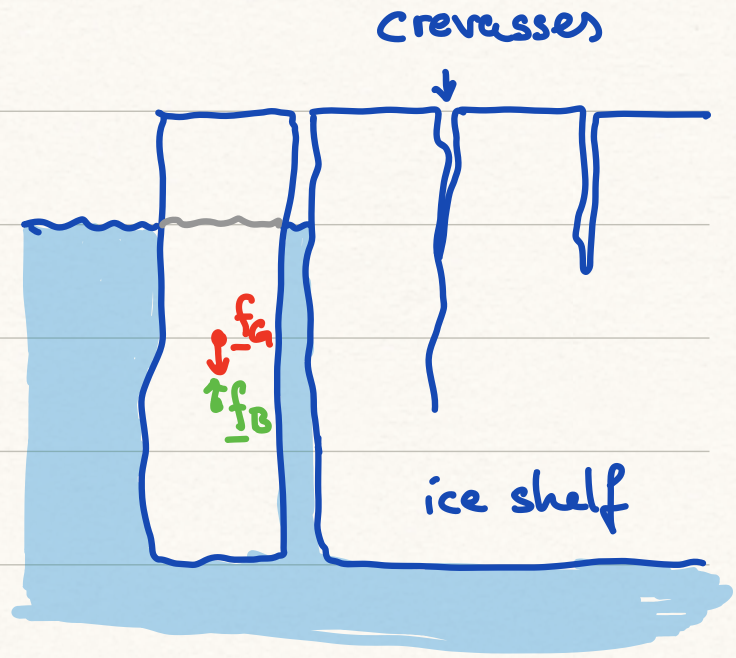
Stability calving fronts

Icebergs formed by calving are often close to rectangular prisms and metastable.



Rectangular Prism

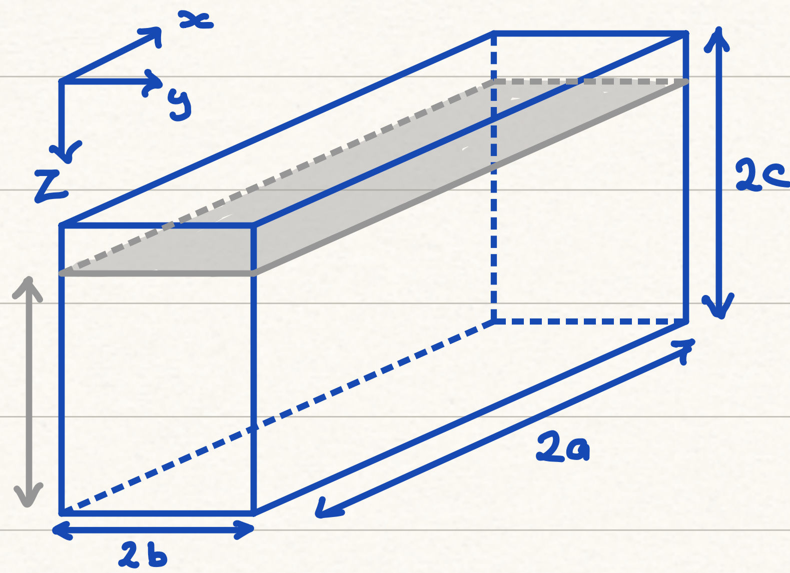
dimension: $2a, 2b, 2c$

draft: d

Total volume: $V_T = 8abc$

Displaced volume: $V_D = 4abd$

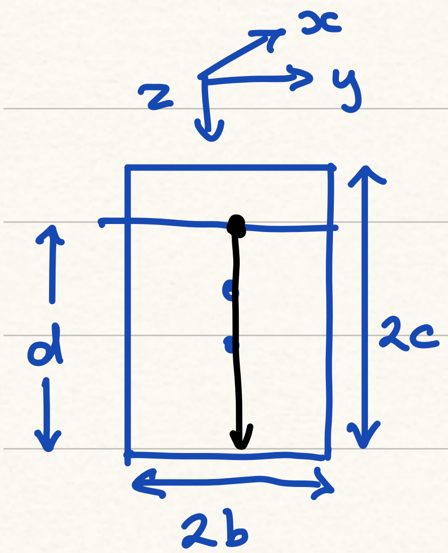
Waterline area: $A = 4ab$



Stability to rotation around x-axis $\Rightarrow$ yz-cross section

Iceberg Stability

Initial Geometry



2D volume / Areas:

Total: $A_T = 4bc$

Buoyancy: $A_B = 2bd$

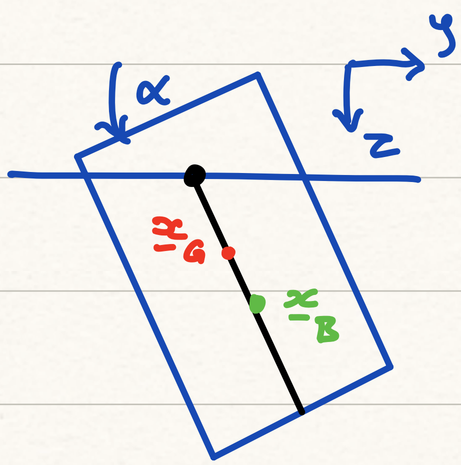
Centroids:

Gravity: $x_G = y_G = 0 \quad z_G = d - z > 0$

Buoyancy: $x_B = y_B = 0 \quad z_B = c > 0$

$\Rightarrow$ meta stable

Rotated Geometry:



counter clockwise

around x-axis: $\alpha > 0$

hydrostatic torque:

$$\underline{\tau}_H = \underbrace{m(\underline{x}_G - \underline{x}_B)}_{\underline{a}} \times \underline{g} \qquad \underline{g} = -g\hat{z}$$

$$\underline{a} \times \underline{g} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ 0 & 0 & -g \end{vmatrix} = \hat{x}(-g a_y - 0) - \hat{y}(-g a_x - 0) + \hat{z}(0 - 0)$$

$\underline{a} \times \underline{g} = -a_y g \hat{x} + a_x g \hat{y}$

substitute into $\underline{\tau}_H$:

$\tau_x = -(y_G - y_B)\, mg$ rotating around $\hat{\underline{x}}$

$\tau_y = +(x_G - x_B)\, mg$ $x_G = x_B = 0$

$\tau_z = 0$

$$\Rightarrow \boxed{\tau_x = -(y_G - y_B)\, mg}$$

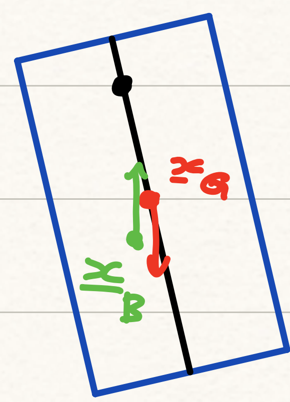
stabilizing: $\tau_x < 0$ (clockwise)

destabilizing: $\tau_x > 0$ (counterclockwise)

(assumes $\alpha > 0$ counterclockwise)

Stability criterium: $(m > 0,\ g > 0)$

$$\boxed{\frac{\tau_x}{mg} = y_B - y_G < 0}$$



horizontal displacement
of center of gravity
exeeds center of buoyancy

Horizontal displacements

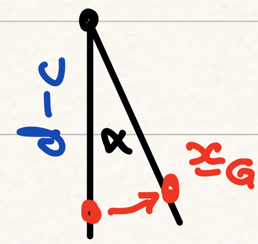
assume $\alpha \ll 1 \quad \Rightarrow \tan\alpha \approx \sin\alpha \approx \alpha$

1) Center of gravity

shape of ice berg remains same

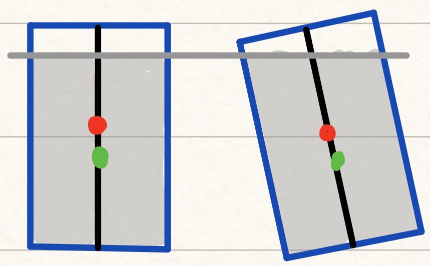
$\Rightarrow$ simple rotation

$$y_G \approx \alpha(d-c) > 0$$



2) Center of buoyancy

shape of submerged body changes



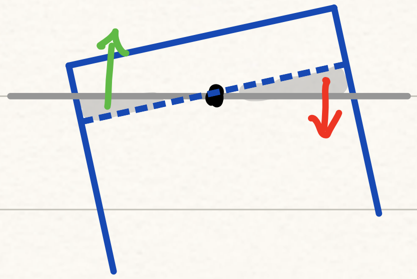
y_B changes due to

1) rotation $\quad \Delta y_B^R$

2) change in shape $\quad \Delta y_B^S$

$$\Rightarrow \quad y_B = \Delta y_B^R + \Delta y_B^S$$

as before: $\Delta y_B^R \approx \alpha \frac{d}{2}$ $\quad$ destabilizing $(z_B > z_G)$

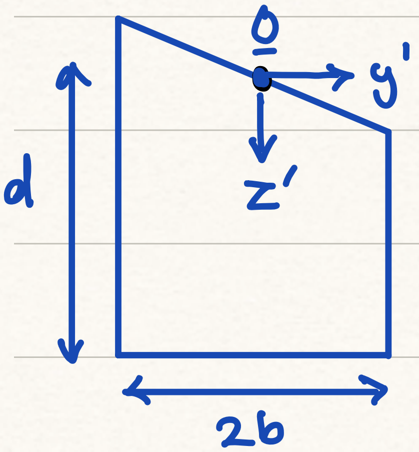


change in shape is stabilizing!

Lateral change due to change in shape

new coordinate system y' & z'

aliged with trapezoid



$A_B = 2bd$ for $\alpha \ll 1: y' \approx y$

$z'_{Top} = \alpha y'$

$z'_{Bot} = d$

y-centroid: $\Delta y_B^S = \frac{1}{A_B} \int_{-b}^{b} \int_{z_{Top}}^{z_{Bot}} y' \, dz' dy'$

$$\int_{z_{Top}}^{z_{Bot}} dz' = z_{Bot} - z_{Top} = d - \alpha y'$$

substitute: $\Delta y_B^S = \frac{1}{A_B} \int_{-b}^{b} y' (d - \alpha y') \, dy'$

$$= \frac{1}{A_B} \left(\frac{d y'^2}{2} - \frac{\alpha y'^3}{3} \right) \Bigg|_{-b}^{b} = -\frac{1}{A_B} \frac{2\alpha b^3}{3}$$

$A_B = 2bd$ $\qquad \Delta y_B^S = -\frac{\alpha b^2}{3d} \Rightarrow$ stabilizing

Vertical centroid: $\frac{d}{2} - \frac{\alpha^2 b^2}{6d} \approx \frac{d}{2} \Rightarrow$ no change!

Total horizontal displacement:

$$y_B = \Delta y_B^R + \Delta y_B^S$$
$$= \frac{\alpha d}{2} - \frac{\alpha b^2}{3d}$$

Substitute into stability criterium:

$$\frac{I_x}{mg} = y_B - y_G < 0 \qquad y_G = \alpha(d-c)$$

$$\frac{\cancel{\alpha} d}{2} - \frac{\cancel{\alpha} b^2}{3d} - \cancel{\alpha}(d-c) < 0$$
$$d^2 - 2cd > -\frac{2}{3}b^2$$

aspect ratios of submerged body: $\frac{b}{c}$ & $\frac{d}{c}$

$$\left(\frac{d}{c}\right)^2 - 2\frac{d}{c} > -\frac{2}{3}\left(\frac{b}{c}\right)^2$$

complete square: $\left(\frac{d}{c}\right)^2 - 2\frac{d}{c} + 1 = \left(\frac{d}{c} - 1\right)^2$

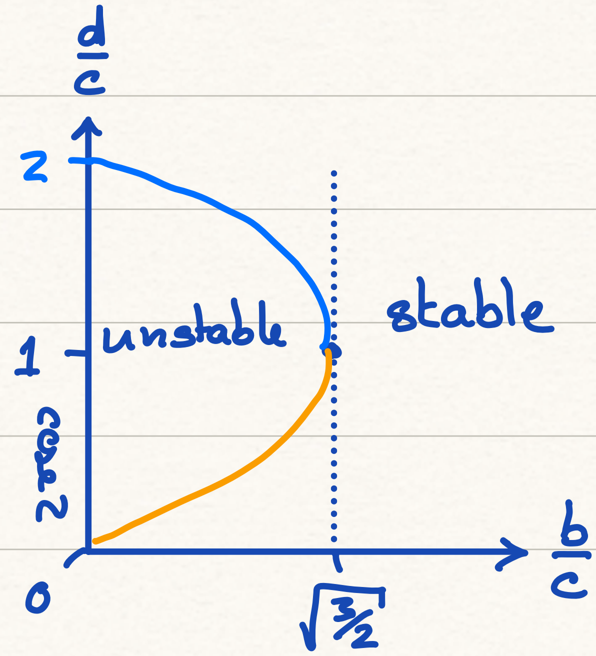
$$\boxed{\left(\frac{d}{c} - 1\right)^2 > 1 - \frac{2}{3}\left(\frac{b}{c}\right)^2}$$

lhs: $\left[\left(\frac{d}{c}\right)^2 - 1\right]^2 \geq 0$

rhs < 0: always stable

$\Rightarrow \quad 1 - \frac{2}{3}\left(\frac{b}{c}\right)^2 < 0$

$\frac{b}{c} > \sqrt{\frac{3}{2}}$



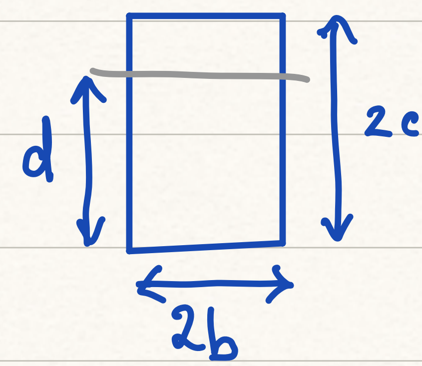
rhs > 0: cond. stable

case 1: $d/c > 1$

$d/c > 1 + \sqrt{1 - \frac{2}{3}(b/c)^2}$

case 2: $d/c < 1$

$d/c < 1 - \sqrt{1 - \frac{2}{3}(b/c)^2}$



Relate $\frac{d}{c}$ to desity contrast wsind isostacy

$\rho_i = 917 \frac{kg}{m^3} \qquad \rho_w = 1025 \frac{kg}{m^3}$

$m_i = m_w$

$\rho_i \, 8 \cancel{ab} c = \rho_w \, 4 \cancel{ab} d \qquad 2 \rho_i c = \rho_w d$

$\Rightarrow \frac{d}{2c} = \frac{\rho_i}{\rho_w} = 0.89 \Rightarrow$ 90% under water

Solve for stabe aspect ratio $\frac{b}{c}$

$$\left(\frac{d}{c}-1\right)^2 = 1 - \frac{2}{3}\left(\frac{b}{c}\right)^2$$

$$\frac{2}{3}\left(\frac{b}{c}\right)^2 = 1 - \left(\frac{d}{c}-1\right)^2$$

$$\left(\frac{b}{c}\right)^2 = \frac{3}{2}\left[1-\left(\frac{d}{c}-1\right)^2\right] =$$

$$= \frac{3}{2}\left[1-\left(\left(\frac{d}{c}\right)^2 - 2\frac{d}{c} + 1\right)\right]$$

$$= \frac{3}{2}\left[\left(\frac{d}{c}\right)^2 + 2\frac{d}{c}\right]$$

$$\boxed{\frac{b}{c} = \sqrt{\frac{3}{2}\left[\left(\frac{d}{c}\right)^2 + 2\frac{d}{c}\right]}}$$

substituting values of ice & water $\frac{d}{c} = 1.8$

$\frac{b}{c} \approx 0.75 = \frac{3}{4}$ $\boxed{\frac{b}{c} \approx \frac{3}{4}}$