Green-Lagrange Strain Tensor

right Cauchy-Green: $\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}}$

$\Rightarrow$ quantifies streches & shear

$$\lambda(\hat{\underline{X}}) = |d\underline{x}| / |d\underline{X}| = \sqrt{\hat{\underline{X}} \cdot \underline{\underline{C}} \hat{\underline{X}}}$$

No deformation: $\underline{x} = \varphi(\underline{X}) = \underline{X} \Rightarrow \underline{\underline{F}} = \underline{\underline{C}} = \underline{\underline{I}} \Rightarrow \lambda = 1$

Constitutive law: $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}(\underline{\underline{C}})$

no deformation = no stress

$\Rightarrow$ Strain tensor zero in absence of deformation

1D: engineering strain: $e = \lambda - 1$

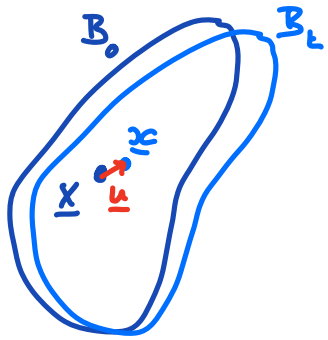
3D Green-Lagrange strain tensor

$$\boxed{\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}})}$$

no deformation: $\underline{\underline{F}} = \underline{\underline{C}} = \underline{\underline{I}} \Rightarrow \underline{\underline{E}} = \underline{\underline{0}}$

relation to infinitesimal strain tensor

Small displacements



natural to use displacement

$$\underline{u} = \underline{x} - \underline{X} \qquad \underline{x} = \varphi(\underline{X})$$

$$\nabla \underline{u} = \nabla(\varphi(\underline{X}) - \underline{X}) = \nabla\varphi - \underline{\underline{I}}$$

$$\nabla \underline{u} = \underline{\underline{F}} - \underline{\underline{I}} \equiv \underline{\underline{H}}$$

Small deformation: $|\nabla \underline{u}| = \sqrt{\nabla \underline{u} : \nabla \underline{u}} = \delta \ll 1$

Linearize Cauchy-Green

$$\underline{\underline{F}} = \underline{\underline{I}} + \nabla \underline{u}$$

$$\begin{aligned} \underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}} &= (\underline{\underline{I}} + \nabla \underline{u})^T (\underline{\underline{I}} + \nabla \underline{u}) \\ &= (\underline{\underline{I}} + \nabla \underline{u}^T)(\underline{\underline{I}} + \nabla \underline{u}) \\ &= \underline{\underline{I}} + \nabla \underline{u} + \nabla \underline{u}^T + \underbrace{\nabla \underline{u}^T \nabla \underline{u}}_{O(\delta^2)} \end{aligned}$$

$$\Rightarrow \quad \boxed{\underline{\underline{C}} \approx \underline{\underline{I}} + \nabla \underline{u} + \nabla \underline{u}^T}$$

Linearize Euler-Lagrange

$$\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}}) = \frac{1}{2}(\underline{\underline{I}} + \nabla\underline{u} + \nabla\underline{u}^T - \underline{\underline{I}}) + O(\delta^2)$$

$$\underline{\underline{E}} \approx \frac{1}{2}(\nabla\underline{u} + \nabla\underline{u}^T)$$

Infinitesimal strain tensor:

$$\boxed{\underline{\underline{\varepsilon}} = \frac{1}{2}(\nabla\underline{u} + \nabla\underline{u}^T)} \qquad \underline{\underline{\varepsilon}} = \mathrm{sym}(\nabla\underline{u})$$

Infinitesimal Stretch & Rotation

Linearize right stretch: (this likely came from Holtzapfel)

$$\underline{\underline{U}} = \sqrt{\underline{\underline{C}}} = (\underline{\underline{I}} + \nabla\underline{u} + \nabla\underline{u}^T + \underbrace{\nabla\underline{u}^T\nabla\underline{u}}_{O(\delta^2)})^{\frac{1}{2}}$$

Note: for $\underline{\underline{A}} = \underline{\underline{A}}^T$ $\quad m \in \mathbb{R}$

$$(\underline{\underline{I}} + \underline{\underline{A}})^m = \underline{\underline{I}} + m\,\underline{\underline{A}} + O(|\underline{\underline{A}}|^2)$$

shown with Taylor expansion in principal frame

here $\underline{\underline{A}} = \nabla\underline{u} + \nabla\underline{u}^T + O(\delta^2)$ and $m = \frac{1}{2}$

$$\Rightarrow \quad \underline{\underline{U}} = \underline{\underline{I}} + \frac{1}{2}(\nabla\underline{u} + \nabla\underline{u}^T) + O(\delta^2) \approx \underline{\underline{I}} + \underline{\underline{\varepsilon}}$$

Similarly $\underline{\underline{V}} = \sqrt{\underline{\underline{F}}\underline{\underline{F}}^T} = \underline{\underline{I}} + \frac{1}{2}(\nabla \underline{u} + \nabla \underline{u}^T) = \underline{\underline{I}} + \underline{\underline{\varepsilon}}$

Note: $\underline{\underline{C}} \approx \underline{\underline{I}} + 2\underline{\underline{\varepsilon}}$, $\underline{\underline{U}} \approx \underline{\underline{I}} + \underline{\underline{\varepsilon}}$, $\underline{\underline{V}} \approx \underline{\underline{I}} + \underline{\underline{\varepsilon}}$

$\Rightarrow$ essentially same Linearization (Why?)

Linearize rotation

$\underline{\underline{R}} = \underline{\underline{F}}\,\underline{\underline{U}}^{-1} = (\underline{\underline{I}} + \nabla \underline{u})(\underline{\underline{I}} + \underline{\underline{\varepsilon}})^{-1}$ $\qquad \underline{\underline{\varepsilon}} = O(\delta)$

$= (\underline{\underline{I}} + \nabla \underline{u})(\underline{\underline{I}} - \underline{\underline{\varepsilon}}) + O(\delta^2)$

$= \underline{\underline{I}} - \frac{1}{2}(\nabla \underline{u} + \nabla \underline{u}^T) + \nabla \underline{u} + O(\delta^2)$

$= \underline{\underline{I}} + \frac{1}{2}(\nabla \underline{u} - \nabla \underline{u}^T) + O(\delta^2)$

$\underline{\underline{R}} \approx \underline{\underline{I}} + \frac{1}{2}(\nabla \underline{u} - \nabla \underline{u}^T)$

$= \underline{\underline{I}} + \underline{\underline{\omega}}$

Infinitesimal Rotation Tensor: $\boxed{\underline{\underline{\omega}} = \frac{1}{2}(\nabla \underline{u} - \nabla \underline{u}^T)}$

$\underline{\underline{\omega}} = \text{skew}(\nabla \underline{u})$ = axial tensor

axis of rotation: $a_j = \frac{1}{2} \epsilon_{mjn} \omega_{mn}$

Infinitesimal rotation $\Rightarrow$ skew matrix?

Linearization of $\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}}$

$\underline{\underline{R}} = \underline{\underline{I}} + \underline{\underline{\omega}}$ and $\underline{\underline{U}} \approx \underline{\underline{I}} + \underline{\underline{\varepsilon}}$

$$\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} = (\underline{\underline{I}} + \underline{\underline{\varepsilon}})(\underline{\underline{I}} + \underline{\underline{\omega}}) = \underline{\underline{I}} + \underline{\underline{\varepsilon}} + \underline{\underline{\omega}} + O(|H^2|)$$

$$\Rightarrow \quad \boxed{\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} \approx \underline{\underline{I}} + \underline{\underline{\varepsilon}} + \underline{\underline{\omega}}}$$

$\underline{\underline{R}}\,\underline{\underline{U}}$: multiplicative; $\underline{\underline{\varepsilon}} + \underline{\underline{\omega}}$: additive

$$\underline{\underline{F}} = \underline{\underline{I}} + \nabla \underline{u} = \underline{\underline{I}} + \mathrm{sym}(\nabla \underline{u}) + \mathrm{skew}(\nabla \underline{u})$$

⇒ nature of the rotation-stretch decomposition changes from multiplicative to additive ∇!

⇒ addition commutes but multiplication does not!

Finite theory: right and left version

Infinitesimal theory no difference.

Interpretation of components of $\underline{\underline{\varepsilon}}$

Start from $\underline{\underline{C}}$

$\underline{\underline{\varepsilon}} = \underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}}) + O(\delta^2)$ $\qquad |\nabla \underline{u}| = \delta$

$\Rightarrow \underline{\underline{C}} \approx \underline{\underline{I}} + 2\underline{\underline{\varepsilon}}$

I) Diagonal components

$C_{ii} = 1 + 2\varepsilon_{ii}$ $\qquad \sqrt{C_{ii}} = \lambda(\underline{e}_i)$

$\lambda(\underline{e}_i) = \sqrt{1 + 2\varepsilon_{ii}}$

Expand in Taylor-Series: $\sqrt{1-x^1} = 1 + \frac{x}{2} - \frac{x^2}{8} + \dots$

where $x = 2\varepsilon_{ii}$ s.t. $\lambda(\underline{e}_i) = 1 + \varepsilon_{ii}$

$\Rightarrow$ $\boxed{\varepsilon_{ii} \approx \lambda(\underline{e}_i) - 1}$ engineering strain in coordinate directions

$\lambda = \frac{|\underline{y} - \underline{x}|}{|\underline{Y} - \underline{X}|} = \frac{\ell}{L}$ stretch

$\lambda - 1 = \frac{|\underline{y} - \underline{x}| - |\underline{Y} - \underline{X}|}{|\underline{Y} - \underline{X}|} = \frac{\ell - L}{L} = \frac{\Delta \ell}{L}$

ε_{ii} = relative change in Length

II) Off-Diagonal Components

$$C_{ij} = \lambda(\underline{e}_i)\,\lambda(\underline{e}_j)\sin(\gamma_{ij}) \qquad \gamma_{ij} = \gamma(\underline{e}_i, \underline{e}_j)$$

$$C_{ij} \approx 2\,\varepsilon_{ij} \qquad i \neq j$$

$$\varepsilon_{ij} \approx \frac{1}{2}\lambda(\underline{e}_i)\,\lambda(\underline{e}_j)\sin(\gamma_{ij})$$

$$\begin{aligned}
\varepsilon_{ij} &= \frac{1}{2}(1+\varepsilon_{ii})(1+\varepsilon_{jj})\sin(\gamma_{ij}) \\
&= \frac{1}{2}\left[1+\varepsilon_{ii}+\varepsilon_{jj}+\varepsilon_{ii}\varepsilon_{jj}\right]\sin(\gamma_{ij}) \\
&= \frac{1}{2}\left[\sin(\gamma_{ij}) + \underbrace{\varepsilon_{ii}\sin(\gamma_{ij})}_{O(\epsilon^2)} + \underbrace{\varepsilon_{jj}\sin(\gamma_{ij})}_{O(\epsilon^2)} + \underbrace{\varepsilon_{ii}\varepsilon_{jj}\sin(\gamma_{ij})}_{O(\epsilon^3)}\right]
\end{aligned}$$

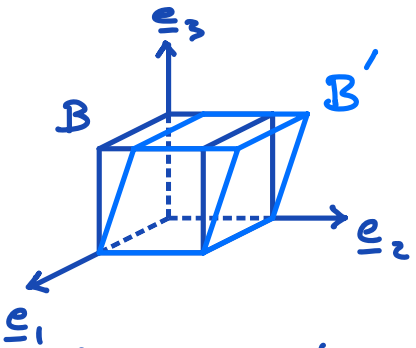
$$\Rightarrow \quad \varepsilon_{ij} = \frac{1}{2}\sin(\gamma(\underline{e}_i, \underline{e}_j))$$

Infinitesimal strain $\Rightarrow \gamma \ll 1 \Rightarrow \sin\gamma \approx \gamma$

$$\boxed{\varepsilon_{ij} \approx \frac{1}{2}\gamma(\underline{e}_i, \underline{e}_j)}$$

$\Rightarrow$ half shear angle between coord. directions

Example: Simple shear



$$\underline{x} = \varphi(\underline{X}) = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 + \alpha X_3 \\ X_3 \end{bmatrix} \qquad \alpha > 0$$

large def: $\underline{\underline{C}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & \alpha & 1+\alpha^2 \end{bmatrix}$

(last lecture)

Infinitesimal:

$$\underline{u} = \underline{x} - \underline{X} = \begin{bmatrix} 0 \\ \alpha X_3 \\ 0 \end{bmatrix} \qquad \nabla \underline{u} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{bmatrix}$$

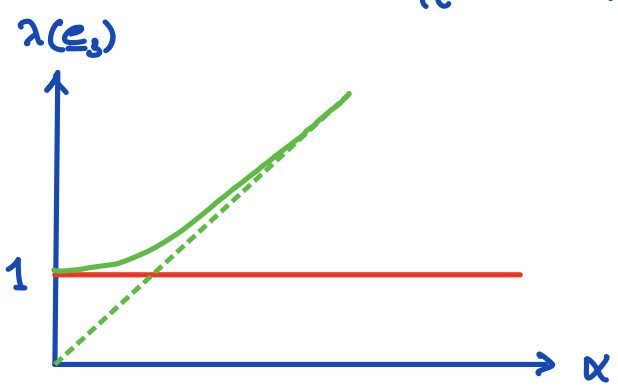
$$\underline{\underline{\varepsilon}} = \frac{1}{2}(\nabla \underline{u} + \nabla \underline{u}^T) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha/2 \\ 0 & \alpha/2 & 0 \end{bmatrix}$$

Infinitesimal stretches: $\varepsilon_{ii} = \lambda(\underline{e}_i) - 1 = 0$

$\lambda(\underline{e}_i) = 1$ no stretch in any coord. direction !

Finite stretches: $C_{ii} = \lambda^2(\underline{e}_i)$ $\quad C_{33} = \lambda^2(\underline{e}_3) = 1+\alpha^2$

$$\lambda(\underline{e}_3) = \sqrt{1+\alpha^2}$$

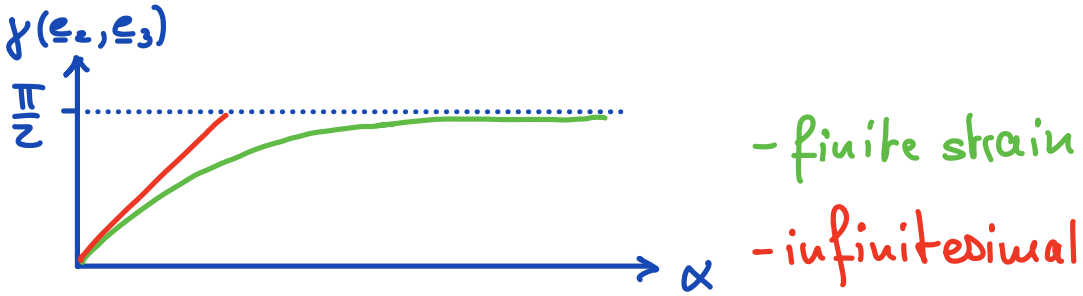


- finite strain
- infinitesimal strain

Infinitesimal shear: $\varepsilon_{ij} \approx \frac{1}{2}\gamma(\underline{e}_i, \underline{e}_j)$

$\gamma(\underline{e}_2, \underline{e}_3) = 2\,\varepsilon_{23} = \alpha$

Finite shear: $\gamma(\underline{e}_2, \underline{e}_3) = \operatorname{asin}\left(\frac{\alpha}{\sqrt{1+\alpha^2}}\right)$



⇒ infinitesimal strain is the linearization of finite strain at zero displacement.

Note: $\lambda(\underline{e}_3)$ is under predicted

$\gamma(\underline{e}_2, \underline{e}_3)$ is over predicted

⇒ $\underline{\underline{\varepsilon}}$ is not a lower bound on $\underline{\underline{E}}$ ∇

Volumetric strain

Volume change: $J = \det(\underline{\underline{F}}) = \frac{dV_x}{dV_X}$

Polar decoup: $\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}}$

$J = \det(\underline{\underline{R}}\,\underline{\underline{U}}) = \cancel{\det(\underline{\underline{R}})}^{1}\det(\underline{\underline{U}})$

$J = \det(\underline{\underline{U}}) = \lambda_1\lambda_2\lambda_3$

λ_p's eigenvalues of $\underline{\underline{U}} \Rightarrow$ princ. streches

$\underline{\underline{\varepsilon}}$ is linearization of $\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}})$

engineering strain: $e = \lambda - 1 = \frac{\ell - L}{L} = \frac{\Delta\ell}{L}$

$\Rightarrow$ engineering volumetric strain:

$$\frac{dV_x - dV_X}{dV_X} = \frac{\Delta V}{V_0} = J - 1 = \underline{\lambda_1\lambda_2\lambda_3 - 1}$$

How does this relate to $\underline{\underline{\varepsilon}}$?

To first order: $\underline{\underline{\varepsilon}} \approx \underline{\underline{U}} - \underline{\underline{I}}$

eigenvectors: $\underline{u}_p$ are same to first order

eigenvalues: $\varepsilon_i = \lambda_i - 1$ to first order

$\uparrow \quad \uparrow$

$\underline{\underline{\varepsilon}} \quad \underline{\underline{U}}$ $\Rightarrow \lambda_i = 1 + \varepsilon_i$

Volume change in ε_i's:

$$\frac{\Delta V}{dV_x} = J(\underline{\underline{U}}) - 1 = \lambda_1 \lambda_2 \lambda_3 - 1$$

$$J - 1 = (1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3) - 1$$

$$= 1 + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + \underbrace{(\varepsilon_1\varepsilon_2 + \varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_3)}_{\text{quadratic}} + \underbrace{\varepsilon_1\varepsilon_2\varepsilon_3}_{\text{cubic}} - 1$$

$$\boxed{J - 1 \approx \varepsilon_1 + \varepsilon_2 + \varepsilon_3}$$ (to first order)

Relate this to $\underline{\underline{\varepsilon}} = \varepsilon_{ij}\, \underline{e}_i \otimes \underline{e}_j$

$\varepsilon_{ii} = \lambda(\underline{e}_i) - 1$ $\quad$ $\varepsilon_{ij} = \frac{1}{2}\sin(\gamma(\underline{e}_i, \underline{e}_j))$

Choose: $\underline{e}_i = \underline{u}_i$ (principal frame)

$\Rightarrow \varepsilon_{ii} = \lambda(\underline{u}_i) - 1 \equiv \lambda_i - 1 = \varepsilon_i$ no sum.

$\varepsilon_{ij} = 0$ no shear between $\underline{u}_i$'s (PST)

$$[\underline{\underline{\varepsilon}}]' = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}$$ diagonal

$$tr(\underline{\underline{\varepsilon}}) = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

$$\frac{\Delta V}{dV_x} = J - 1 = tr(\underline{\underline{\varepsilon}})$$

evaluate trace:

$$\underline{\underline{\varepsilon}} = \frac{1}{2}(\nabla \underline{u} + \nabla \underline{u}^T)$$

$$tr(\underline{\underline{\varepsilon}}) = \frac{1}{2}\left(tr(\nabla \underline{u}) + tr(\nabla \underline{u}^T)\right) \qquad tr(\nabla \underline{u}^T) = tr(\nabla \underline{u})$$

$$= tr(\nabla \underline{u}) = \nabla \cdot \underline{u}$$

$$\boxed{\frac{dV_x - dV_x}{dV_x} = \frac{\Delta V}{V_0} = J - 1 = tr(\underline{\underline{\varepsilon}}) = \nabla \cdot \underline{u}}$$

Link between volume change and divergence of displacement!

Finite strain: $J = \det(\underline{\underline{F}})$

Infinitesimal strain: $\nabla \cdot \underline{u}$

Trace of $\underline{\underline{\varepsilon}}$ gives relative change in volume.

$\Rightarrow$ split $\underline{\underline{\varepsilon}}$ into volumetric & deviatoric parts

$$\underline{\underline{A}} = \text{sph}(\underline{\underline{A}}) + \text{dev}(\underline{\underline{A}})$$

$$\text{sph}(\underline{\underline{A}}) = \frac{1}{3}\,\text{tr}(\underline{\underline{A}})\,\underline{\underline{I}}$$

$$\text{dev}(\underline{\underline{A}}) = \underline{\underline{A}} - \text{sph}(\underline{\underline{A}})$$

$$\text{sph}(\underline{\underline{\varepsilon}}) = \frac{1}{3}\,\text{tr}(\underline{\underline{\varepsilon}})\,\underline{\underline{I}} = \frac{1}{3}(\nabla\cdot\underline{u})\,\underline{\underline{I}}$$

$$\text{dev}(\underline{\underline{\varepsilon}}) = \underline{\underline{\varepsilon}}' = \underline{\underline{\varepsilon}} - \text{sph}(\underline{\underline{\varepsilon}})$$

$$\boxed{\underline{\underline{\varepsilon}}' = \frac{1}{2}(\nabla\underline{u} + \nabla\underline{u}^T) - \frac{1}{3}(\nabla\cdot\underline{u})\,\underline{\underline{I}}}$$

spherical strain: volume change

deviatoric strain: shape change