Extremal Shear Stresses

Last time $\longrightarrow$ extremal normal str

normal stress: $\sigma_n = \underline{n} \cdot \underline{\underline{\sigma}} n$ (quadratic, n_i^2)

shear stress: $|\underline{t}|^2 = \sigma_n^2 + \tau^2$ (quartic, n_i^4)

$\Rightarrow$ algebra gets messy.

Mohr circle

$\Rightarrow$ graphical & intuitive

In principal frame we have 3 equations:

1) $\sigma_n = \underline{n} \cdot \underline{\underline{\sigma}}\, \underline{n}$ normal stress

2) $|\underline{t}|^2 = \tau^2 + \sigma_n^2$ Pythagoras

3) $\underline{n} \cdot \underline{n} = 1$ unit vector

In terms of components:

1) $\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 = \sigma_n$

2) $\sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 = \underbrace{\tau^2 + \sigma_n^2}_{q}$

3) $n_1^2 + n_2^2 + n_3^2 = 1$

$\Rightarrow$ linear system for $x = n_1^2$ $y = n_2^2$ $z = n_3^2$

Linear system

$$\underbrace{\begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_1^2 & \sigma_2^2 & \sigma_3^2 \\ 1 & 1 & 1 \end{pmatrix}}_{\underline{A}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \sigma_n \\ 9 \\ 1 \end{pmatrix}$$

$$\underline{x} = \underline{b}$$

Solve using Cramers rule:

$\underline{\underline{A}}\,\underline{x} = \underline{b}$ $\qquad$ $\det A \neq 0$ $\qquad$ $\underline{\underline{A}}$ is $n \times n$

$$x_k = \frac{\det \underline{\underline{A}}_k}{\det \underline{\underline{A}}} \qquad k = 1, \dots, n$$

$\underline{\underline{A}}_k$ is $\underline{\underline{A}}$ with k-th column replaced by $\underline{b}$.

$\Rightarrow$ this will give clean answer.

$$\det(\underline{\underline{A}}) = \det(\underline{\underline{A}}^T) = \begin{vmatrix} \sigma_1 & \sigma_1^2 & 1 \\ \sigma_2 & \sigma_2^2 & 1 \\ \sigma_3 & \sigma_3^2 & 1 \end{vmatrix} \overset{\text{cyclic perm.}}{=} \begin{vmatrix} 1 & \sigma_1 & \sigma_1^2 \\ 1 & \sigma_2 & \sigma_2^2 \\ 1 & \sigma_3 & \sigma_3^2 \end{vmatrix}$$

$\Rightarrow$ Vandermonde matrix $\qquad \underline{\underline{A}} = \underline{\underline{V}}(\sigma_1, \sigma_2, \sigma_3)$

$$\det \underline{\underline{V}} = (\sigma_2 - \sigma_1)(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)$$

$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$

$$\det \underline{\underline{A}} = -(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)(\sigma_2 - \sigma_3)$$

$\Rightarrow$ as long as $\sigma_1, \sigma_2, \sigma_3$ are distinct $\det A \neq 0$

Apply Cramer's rule:

- $x_1 = n_1^2 = \dfrac{\det \underline{\underline{A}}_1}{\det \underline{\underline{A}}}$ $\qquad \underline{\underline{A}}_1 = \begin{pmatrix} \sigma_n & \sigma_2 & \sigma_3 \\ q & \sigma_2^2 & \sigma_3^2 \\ 1 & 1 & 1 \end{pmatrix}$

$$\det(\underline{\underline{A}}_2) = -(\sigma_2 - \sigma_3)\left[\tau^2 + (\sigma_n - \sigma_2)(\sigma_n - \sigma_3)\right]$$

$$\Rightarrow n_1^2 = \frac{\tau^2 + (\sigma_n - \sigma_2)(\sigma_n - \sigma_3)}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)}$$

- $x_2 = n_2^2 = \dfrac{\det \underline{\underline{A}}_2}{\det \underline{\underline{A}}}$ $\qquad \underline{\underline{A}}_2 = \begin{pmatrix} \sigma_1 & \sigma_n & \sigma_3 \\ \sigma_1^2 & q & \sigma_3^2 \\ 1 & 1 & 1 \end{pmatrix}$

$$\det(\underline{\underline{A}}_2) = (\sigma_1 - \sigma_3)\left[\tau^2 + (\sigma_n - \sigma_1)(\sigma_n - \sigma_3)\right]$$

$$\Rightarrow n_2^2 = -\frac{\tau^2 + (\sigma_n - \sigma_1)(\sigma_n - \sigma_3)}{(\sigma_2 - \sigma_3)(\sigma_1 - \sigma_2)}$$

- $x_3 = n_3^2 = \dfrac{\det \underline{\underline{A}}_3}{\det \underline{\underline{A}}}$ $\qquad \underline{\underline{A}}_3 = \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_n \\ \sigma_1^2 & \sigma_2^2 & q \\ 1 & 1 & 1 \end{pmatrix}$

$$\det \underline{\underline{A}}_3 = -(\sigma_1 - \sigma_2)\left[\tau^2 + (\sigma_n - \sigma_1)(\sigma_n - \sigma_2)\right]$$

$$\Rightarrow \quad n_3^2 = \frac{\tau^2 + (\sigma_n - \sigma_1)(\sigma_n - \sigma_2)}{(\sigma_1 - \sigma_3)\,(\sigma_2 - \sigma_3)}$$

Summary:

$$\boxed{\begin{aligned} n_1^2 &= \frac{\tau^2 + (\sigma_n - \sigma_2)(\sigma_n - \sigma_3)}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)} \\[2ex] n_2^2 &= -\frac{\tau^2 + (\sigma_n - \sigma_1)(\sigma_n - \sigma_3)}{(\sigma_2 - \sigma_3)(\sigma_1 - \sigma_2)} \\[2ex] n_3^2 &= \frac{\tau^2 + (\sigma_n - \sigma_1)(\sigma_n - \sigma_2)}{(\sigma_1 - \sigma_3)\,(\sigma_2 - \sigma_3)} \end{aligned}}$$

Sanity check: $\sigma_n = \sigma_1 \qquad \tau = 0$

$$\left.\begin{aligned} n_1^2 &= \frac{0^2 + (\sigma_1 - \sigma_3)(\sigma_1 - \sigma_2)}{(\sigma_1 - \sigma_3)(\sigma_1 - \sigma_2)} = 1 \quad \checkmark \\ n_2^2 &= \cancelto{0}{(\sigma_1 - \sigma_1)}(\sigma_1 - \sigma_3) / [(\sigma_2 - \sigma_3)(\sigma_1 - \sigma_2)] = 0 \\ n_3^2 &= \cancelto{0}{(\sigma_1 - \sigma_1)} \ \ldots \ = 0 \end{aligned}\right\} \quad \underline{n} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Non-Negativity constraints: $n_i^2 \geq 0$

denominators are positive

3 inequalities:

1) $\tau^2 + (\sigma_n - \sigma_2)(\sigma_n - \sigma_3) \geq 0$

2) $\tau^2 + (\sigma_n - \sigma_1)(\sigma_n - \sigma_3) \leq 0$ (minus sign)

3) $\tau^2 + (\sigma_n - \sigma_1)(\sigma_n - \sigma_2) \geq 0$

Note: n_i's have been eliminated

Dependence of σ_n & τ on principal stresses

Completing squares:

$$(\sigma_n - p)(\sigma_n - q) = \left(\sigma_n - \frac{p+q}{2}\right)^2 - \left(\frac{p-q}{2}\right)^2$$

Mohr's circles

Circle 1 ($n_1^2 \geq 0$):

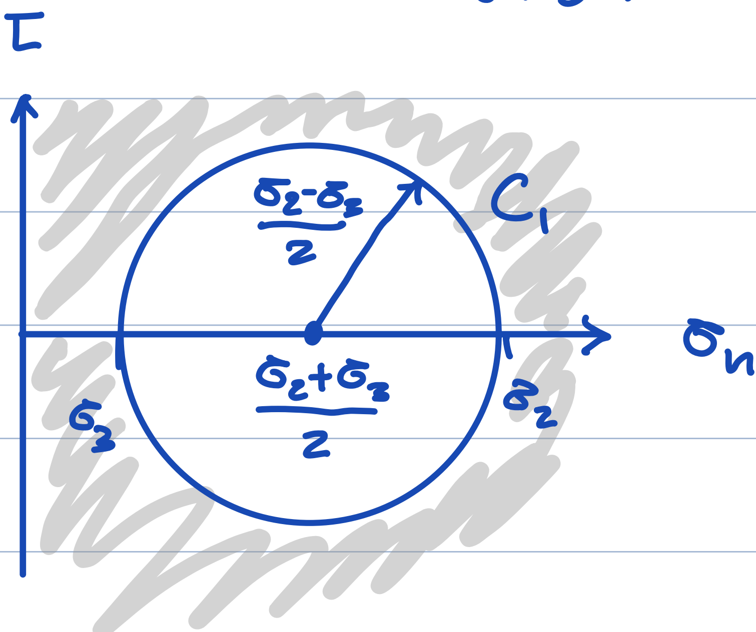
$$\tau^2 + (\sigma_n - \sigma_2)(\sigma_n - \sigma_3) > 0$$

$$\tau^2 + \left(\sigma_n - \frac{\sigma_2+\sigma_3}{2}\right)^2 - \left(\frac{\sigma_2-\sigma_3}{2}\right)^2 \geq 0$$

$$\boxed{\tau^2 + \left(\sigma_n - \frac{\sigma_2+\sigma_3}{2}\right)^2 \geq \left(\frac{\sigma_2-\sigma_3}{2}\right)^2}$$

Eqn of circle: $(x - x_0)^2 + (y - y_0)^2 = R^2$

(x_0, y_0) = center $\qquad R$ = radius



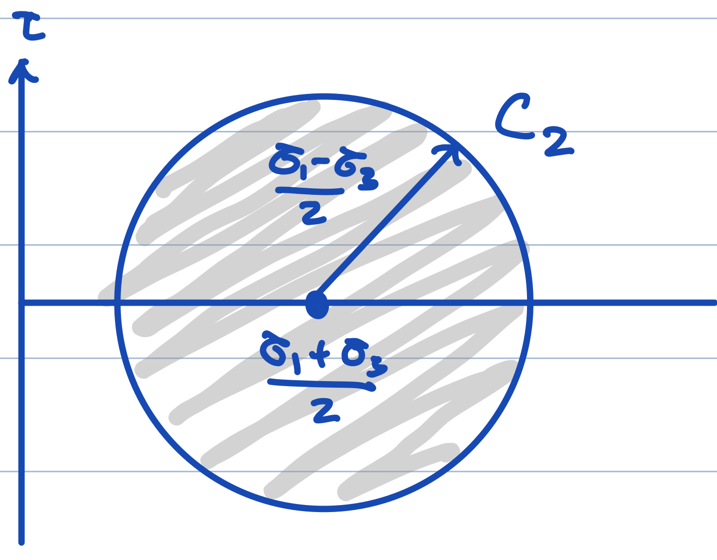
admissible τ & σ_n are outside C_1

Circle 2 $(n_2^2 \geq 0)$:

$$\tau^2 + (\sigma_n - \sigma_1)(\sigma_n - \sigma_3) \leq 0$$

$$\tau^2 + \left(\sigma_n - \frac{\sigma_1 + \sigma_3}{2}\right)^2 - \left(\frac{\sigma_1 - \sigma_3}{2}\right)^2 \leq 0$$

$$\boxed{\tau^2 + \left(\sigma_n - \frac{\sigma_1 + \sigma_3}{2}\right)^2 \leq \left(\frac{\sigma_1 - \sigma_3}{2}\right)^2}$$



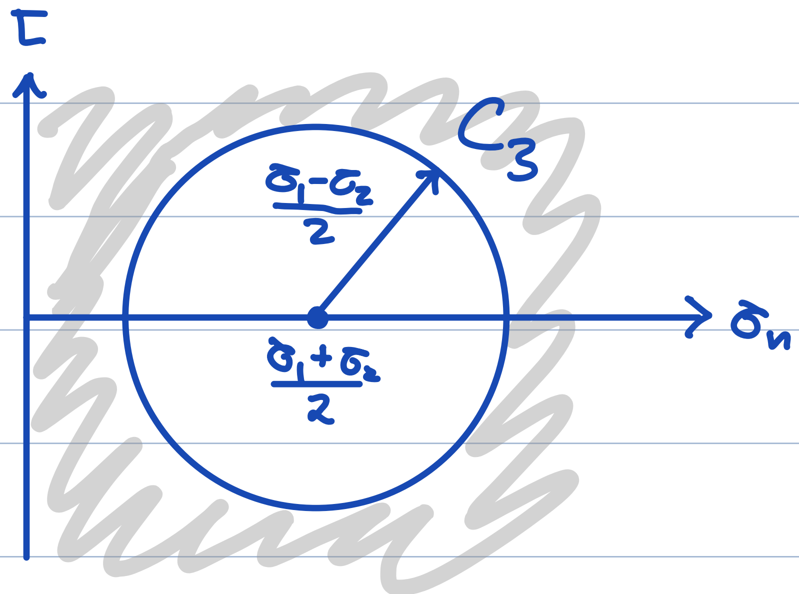
admissible τ & σ_n are inside circle

Circle 3 ($n_3^2 \geq 0$)

$$\tau^2 + (\sigma_n - \sigma_1)(\sigma_n - \sigma_2) \geq 0$$

$$\tau^2 + \left(\sigma_n - \frac{\sigma_1+\sigma_2}{2}\right)^2 - \left(\frac{\sigma_1-\sigma_2}{2}\right)^2 \geq 0$$

$$\boxed{\tau^2 + \left(\sigma_n - \frac{\sigma_1+\sigma_2}{2}\right)^2 \geq \left(\frac{\sigma_1-\sigma_2}{2}\right)^2}$$

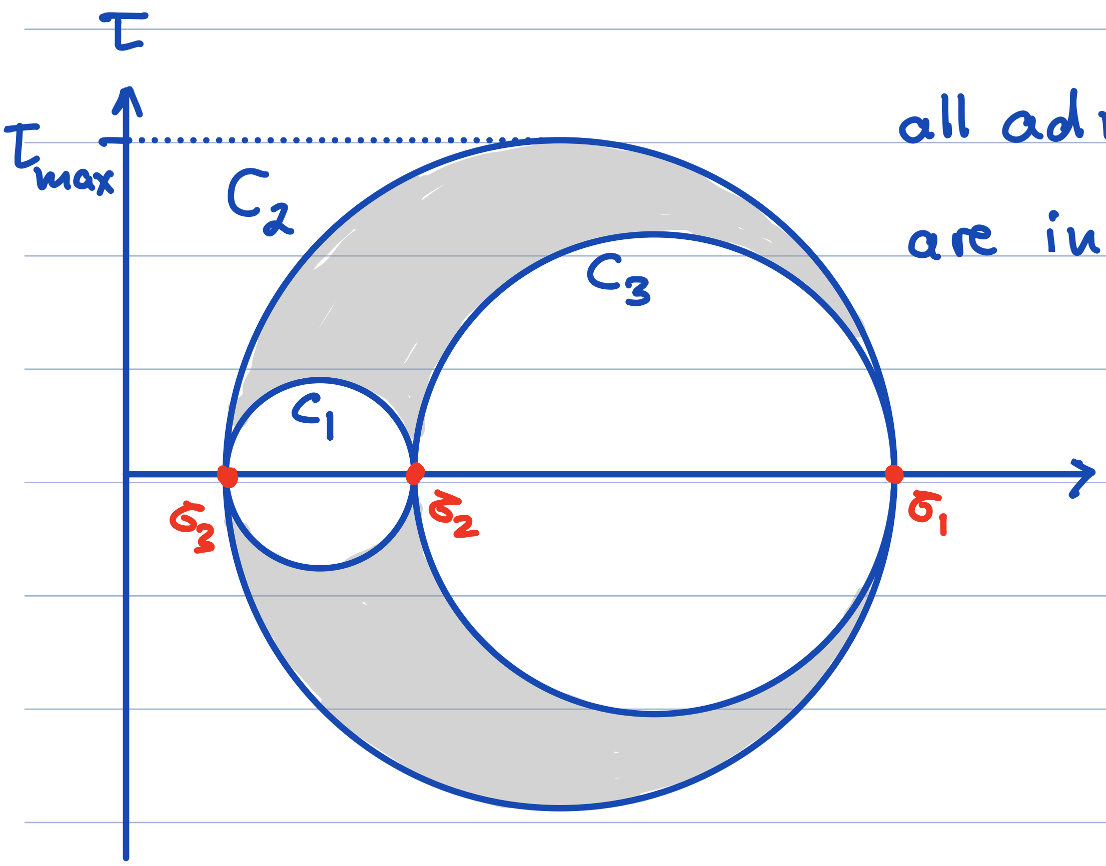


admissible τ & σ_n are outside circle

Combine 3 different conditions

$\Rightarrow$ 3D Mohr circle

3 D Mohr circle



all admissible τ & σ_n are in gray area

Sanity check: $\sigma_n = \sigma_i \rightarrow \tau = 0$

no shear stress on principal planes

max shear stress determined by C_2

$$\tau^2 + \left(\sigma_n - \frac{\sigma_1 + \sigma_3}{2}\right)^2 \leq \Bigg(\underbrace{\frac{\sigma_1 - \sigma_3}{2}}_{\text{radius}}\Bigg)^2$$

$$\boxed{\tau_{max} = \pm \frac{\sigma_1 - \sigma_3}{2}}$$

corresponding $\sigma_n = \dfrac{\sigma_1 + \sigma_3}{2} \neq 0$

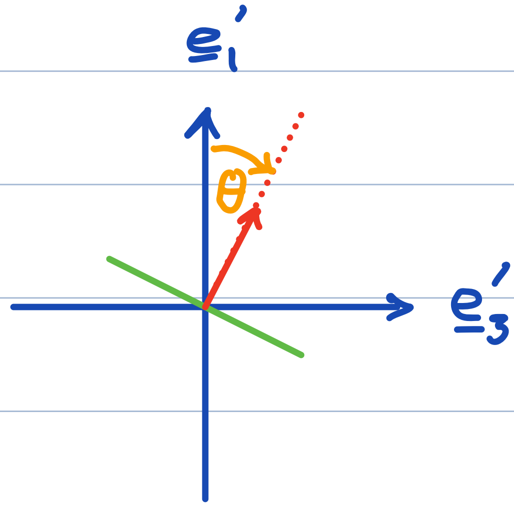
What is the orientation?

$C_2 \rightarrow n_2^2 \geq 0$ on C_2 $n_2 = 0$.

$\Rightarrow n_1^2 + n_3^2 = 1$

$n_1 = \cos\theta$

$n_3 = \sin\theta$



substitute into σ_n:

$$\sigma_n = \sigma_1 n_1^2 + \sigma_3 n_3^2 = \sigma_1 \cos^2\theta + \sigma_3 \sin^2\theta$$

double angle identies: $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\Rightarrow \sigma_n = \underbrace{\frac{\sigma_1+\sigma_3}{2}}_{\text{center}} + \underbrace{\frac{\sigma_1-\sigma_3}{2}}_{\text{radius}} \cos 2\theta$$

substitute into τ:

$$\tau^2 = \sigma_1^2 n_1^2 + \sigma_2 n_2^2 - \sigma_n^2$$

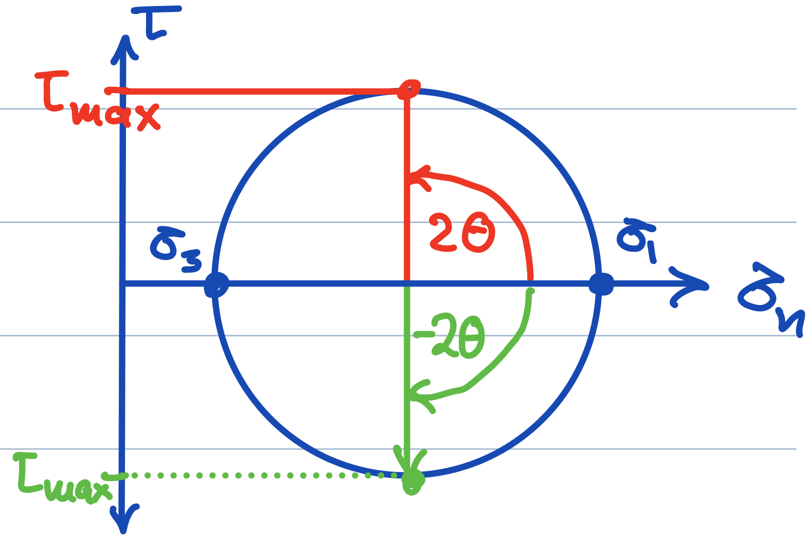
messy algebra utilizing $\cos^2\theta + \sin^2\theta = 1$

$$\tau = \frac{\sigma_1 - \sigma_3}{2} \sin 2\theta$$

Summary

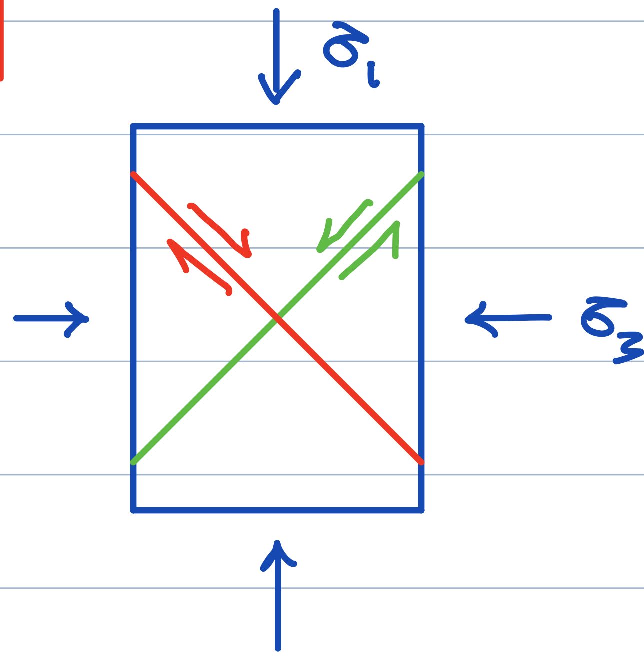
$$\sigma_n = C_2 + R_2 \sin 2\theta$$

$$\tau = R_2 \sin 2\theta$$



$\Rightarrow \boxed{\tau_{max} \text{ at } \pm 45° \text{ to } \underline{e}'_1}$

Do materials fail on plane with maximum $|\tau|$?



Empirical criteria for shear failure

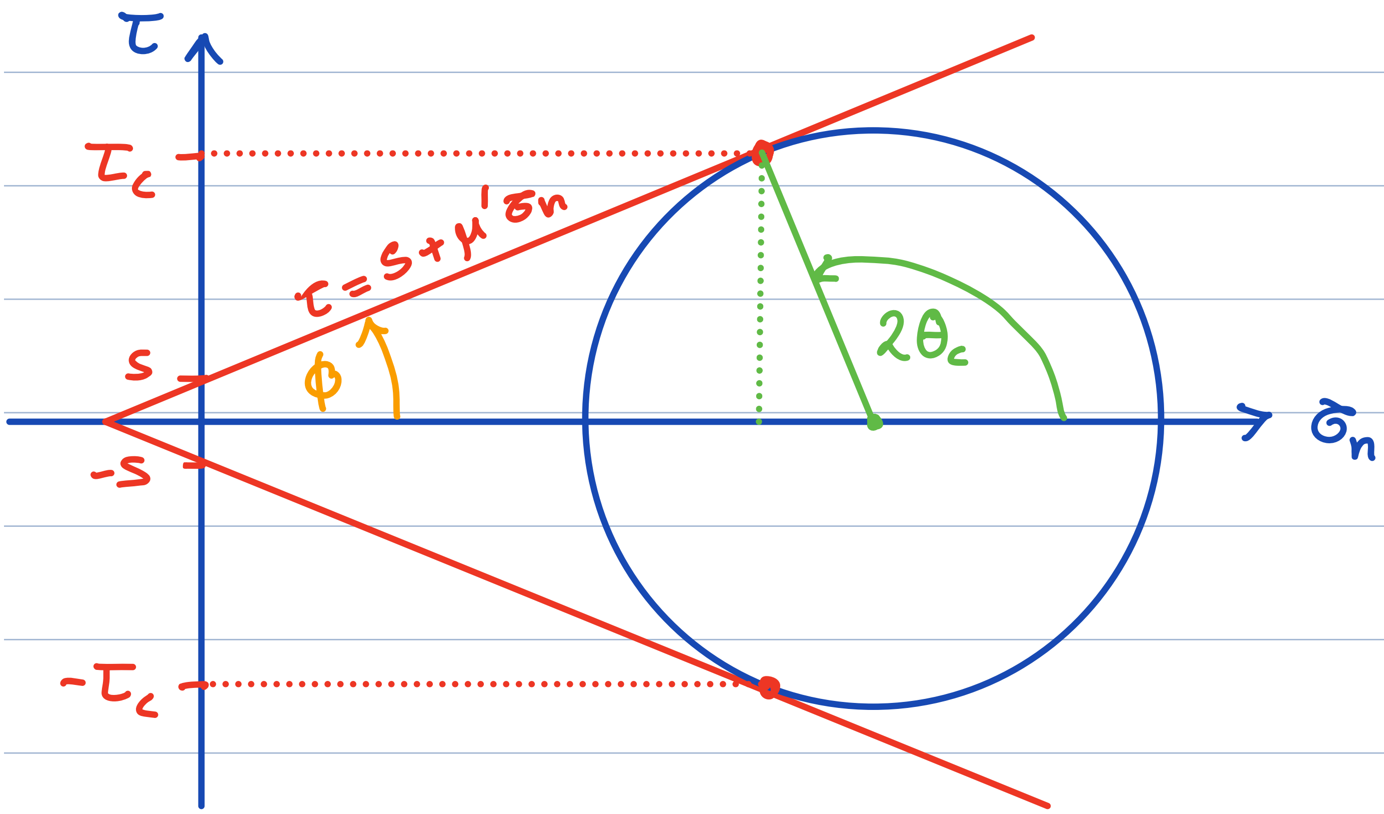
Coulomb failure envelope: $\boxed{|\tau| = S + \mu' \sigma_n}$

S = cohesive strength $\sim 10\text{-}100$ MPa

$\mu' = \tan\phi$ internal friction ~ 0.6

$\phi \approx 30°$ angle of internal friction

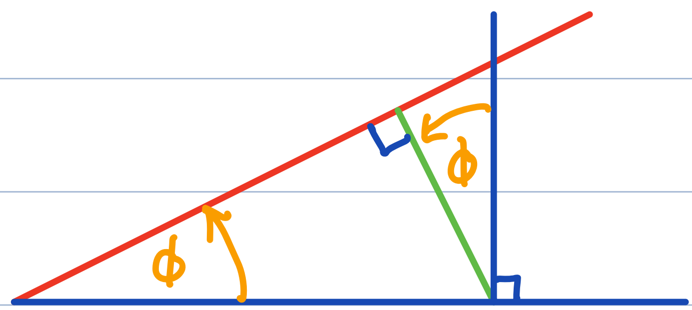
Mohr-Coulomb Failure



$\Rightarrow$ failure occurs at $\tau_c < \tau_{max}$

Angle of failure?

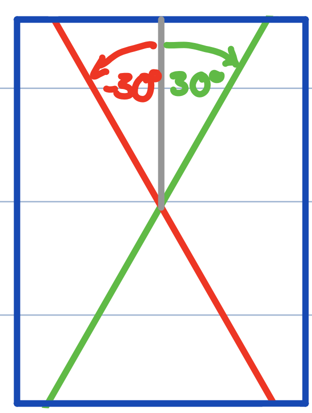
$2\theta_c = 90° + \phi$



$\Rightarrow \boxed{\theta_c = 45° + \frac{\phi}{2}}$

$\phi \approx 30°$

$\theta_c = 45° + 15° = \underline{\underline{60°}}$



$\Rightarrow$ plane is $\pm 30°$ from $\underline{e}'_1$

Here we just use Mohr circle to derive maximum shear stress and Mohr-Coulomb to find failure plane.

There is much more to say:

→ GEO 428 Structural Geology

→ GEO 322M Basin Geomechanics

→ PGE 334 Reservoir Geomechanics