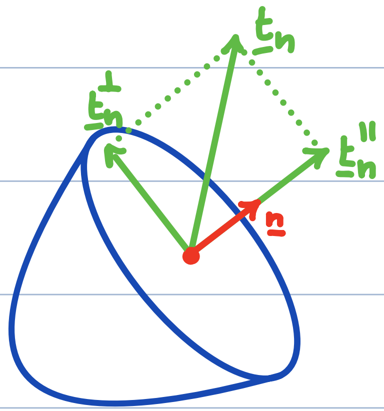
Normal and Shear Stresses



Consider an arbitrary surface in B with normal $\underline{n}$.

$$\underline{\underline{P}}^{\parallel} = \underline{n} \otimes \underline{n} \qquad \underline{\underline{P}}^{\perp} = \underline{\underline{I}} - \underline{n} \otimes \underline{n}$$

normal traction: $\underline{t}^{\parallel} = (\underline{t} \cdot \underline{n})\, \underline{n} = \underline{\underline{P}}^{\parallel}\, \underline{t} = (\underline{n} \otimes \underline{n})\, \underline{t}$

shear traction: $\underline{t}^{\perp} = \underline{t} - \underline{t}^{\parallel} = \underline{\underline{P}}^{\perp}\, \underline{t} = (\underline{\underline{I}} - \underline{n} \otimes \underline{n})\, \underline{t} = -(\underline{t} \times \underline{n}) \times \underline{n}$

The magnitudes of these tractions are

normal stress: $\sigma_n = |\underline{t}^{\parallel}|$

shear stress: $\tau = |\underline{t}^{\perp}|$

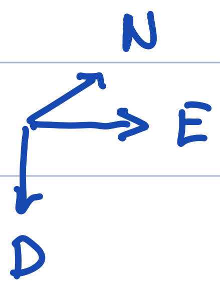
$\sigma_n > 0 \rightarrow$ compressive stress

$\sigma_n < 0 \rightarrow$ tensile stress

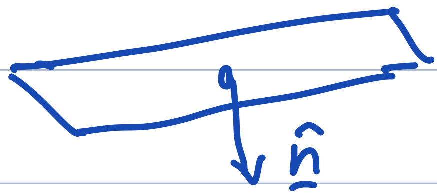
Example: North. Hikurangi subduction zone

(Ema Parker) IODP Expedition 372 & 375

Borehole U1518B $\rightarrow$ Papaku Fault



Fault normal: $\hat{\underline{n}} = \begin{bmatrix} 0.104 \\ -0.18 \\ -0.978 \end{bmatrix} \begin{matrix} N \\ E \\ D \end{matrix}$



Stress on fault: $\underline{\underline{\sigma}} = \begin{bmatrix} 26 & 0.5 & 0 \\ 0.5 & 17 & 0 \\ 0 & 0 & 11 \end{bmatrix}$ MPa

traction on fault:

$$\underline{t} = \underline{\underline{\sigma}}\,\underline{n} = \begin{bmatrix} 26 & 0.5 & 0 \\ 0.5 & 17 & 0 \\ 0 & 0 & 11 \end{bmatrix} \begin{bmatrix} 0.104 \\ -0.18 \\ -0.978 \end{bmatrix} = \begin{bmatrix} 2.61 \\ -3.01 \\ -10.76 \end{bmatrix} \text{ MPa}$$

$$\underline{t}'' = (\underline{n}\cdot\underline{t})\,\underline{n} = \begin{bmatrix} 1.18 \\ -2.04 \\ -11.09 \end{bmatrix} \text{MPa} \qquad \sigma_n = |\underline{t}''| = 11.34 \text{ MPa}$$

$$\underline{t}^{\perp} = \underline{t} - \underline{t}'' = \begin{bmatrix} 1.44 \\ -0.97 \\ 0.33 \end{bmatrix} \text{Mpa} \qquad \tau = |\underline{t}^{\perp}| = 1.76 \text{ MPa}$$

Principal Stresses

Find planes with max. & min. normal stress.

$\sigma_n = |t^n| = \underline{n} \cdot \underline{t} = \underline{n} \cdot \underline{\underline{\sigma}}\, \underline{n}$

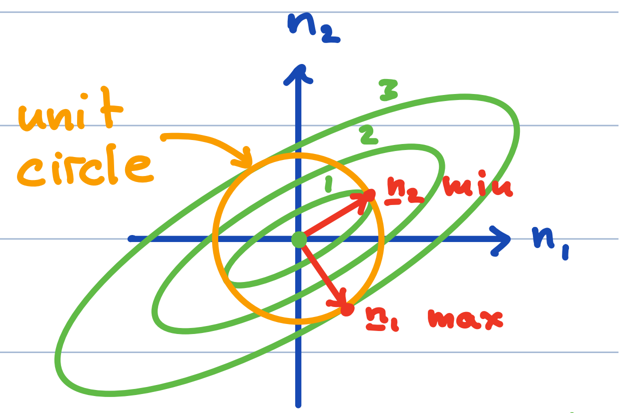
$\underline{n}$ = normal

$$\boxed{\sigma_n(\underline{n}) = \underline{n} \cdot \underline{\underline{\sigma}}\, \underline{n}}$$

find $\max(\sigma_n)$ & $\min(\sigma_n)$

constraint: $|\underline{n}| = \underline{n} \cdot \underline{n} = 1$

$\Rightarrow$ constrained optimization



$\sigma_n(\underline{n}) = \underline{n} \cdot \underline{\underline{\sigma}}\, n$ is quadratic

Note: Contours are ellipses if eigenvalues of $\underline{\underline{\sigma}}$ are positive!

Lagrange multiplier method

$$\boxed{\begin{aligned} \mathcal{L}(\underline{n}, \lambda) &= \underline{n} \cdot \underline{\underline{\sigma}}\, \underline{n} - \lambda(\underline{n} \cdot \underline{n} - 1) \\ \mathcal{L}(n_i, \lambda) &= \underbrace{n_i \sigma_{ij} n_j}_{\text{function}} - \underset{\text{Lagrange multiplier}}{\lambda}\underbrace{(n_i n_i - 1)}_{\text{constraint}} \end{aligned}}$$

stationary points of $\mathcal{L}(\underline{n}, \lambda)$

$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \qquad \frac{\partial \mathcal{L}}{\partial n_k} = 0$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = n_i n_i - 1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial n_k} = \sigma_{ij} (n_{i,k} n_j + n_i n_{j,k}) - \lambda (2 n_i n_{i,k}) = 0$$

where $n_{i,k} = \delta_{ik}$ $\quad n_{j,k} = \delta_{jk}$

$$= \sigma_{ij} (\delta_{ik} n_j + \delta_{jk} n_i) - \lambda (2 n_i \delta_{ik})$$

$$= \sigma_{kj} n_j + \sigma_{ik} n_i - 2 \lambda n_k \qquad \sigma_{ik} = \sigma_{ki} \quad (\underline{\underline{\sigma}}^T = \underline{\underline{\sigma}})$$

$$= \sigma_{kj} n_j + \sigma_{ki} n_i - 2 \lambda n_k \qquad \text{rename } i - j$$

$$= \sigma_{kj} n_j + \sigma_{kj} n_j - 2 \lambda n_k$$

$$= \cancel{2} (\sigma_{kj} n_j - \lambda n_k) = 0 \qquad n_k = \delta_{kj} n_j$$

$$= (\sigma_{kj} - \lambda \delta_{kj}) n_j \qquad \text{remove free index}$$

$$= (\sigma_{kj} - \lambda \delta_{kj}) n_j e_k$$

$$= (\underline{\underline{\sigma}} - \lambda \underline{\underline{I}}) \underline{n}$$

Condition for stationary point

$$\boxed{(\underline{\underline{\sigma}} - \lambda \underline{\underline{I}}) \underline{n} = \underline{0} \quad \text{and} \quad |\underline{n}| = 1}$$

$\Rightarrow$ eigenvalue problem !

What does it mean?

$\underline{n} \cdot (\underline{\underline{\sigma}} - \lambda \underline{\underline{I}}) \underline{n} = 0$

$\underbrace{\underline{n} \cdot \underline{\underline{\sigma}} \, n}_{\sigma_n} - \lambda \underbrace{\underline{n} \cdot \underline{n}}_{1} = 0 \quad \Rightarrow \quad \boxed{\sigma_n = \lambda}$

eigenvalues $\rightarrow$ normal stresses

note : λ_i, $\underline{n}_i$ three eigenvalues & vectors

change notation : $\lambda_i = \sigma_i \quad \underline{n}_i = \underline{v}_i$

$\Rightarrow \quad (\underline{\underline{\sigma}} - \sigma_i \underline{\underline{I}}) \underline{v}_i = 0$

$\underset{\text{max}}{\underset{\uparrow}{\sigma_1}} > \sigma_2 > \underset{\text{min}}{\underset{\uparrow}{\sigma_3}}$: principal normal stresses

$\underline{v}_i$: principal directions of $\underline{\underline{\sigma}}$

$\underline{\underline{\sigma}}^T = \underline{\underline{\sigma}} \quad \Rightarrow \quad \{\underline{v}_i\}$ form principal frame

(orthonormal)

always choose right handed

Cauchy stress in principal frame

$$\boxed{\underline{\underline{\sigma}} = \sum_{i=1}^{3} \sigma_i \, \underline{v}_i \otimes \underline{v}_i = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}}$$

Compare to $\{\underline{e}_i\}$

$$\underline{\underline{\sigma}} = \sigma_{ij} \, \underline{e}_i \otimes \underline{e}_j$$

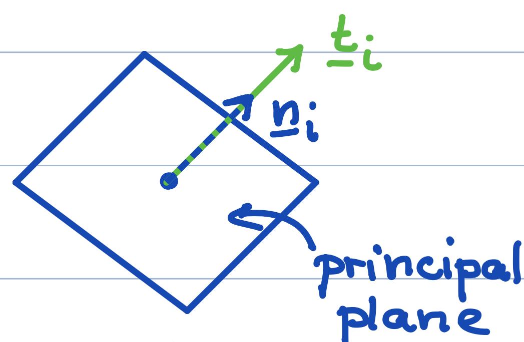
Traction on principal directions $\underline{n} = \underline{v}_i$

$$\underline{t} = \underline{\underline{\sigma}} \, \underline{v}_i = \sum_{k=1}^{3} \sigma_k (\underline{v}_k \otimes \underline{v}_k) \underline{v}_i$$

$$= \sum_{k=1}^{3} \sigma_k (\underline{v}_k \cdot \underline{v}_i) \, \underline{v}_k$$

$$= \sigma_k \, \delta_{ki} \, \underline{v}_k$$

$$= \sigma_i \, \underline{v}_i \quad \text{no summation!}$$



$$\underline{t}^{\parallel} = (\underline{t} \cdot \underline{v}_i) \, \underline{v}_i \qquad \text{no summation}$$

$$= \sigma_i (\underline{v}_i \cdot \underline{v}_i) \, \underline{v}_i = \sigma_i \, \underline{v}_i = \underline{t}$$

$$\Rightarrow \sigma_n = \sigma_i$$

$$\underline{t}^{\perp} = \underline{t} - \underline{t}^{\parallel} = \underline{0} \qquad \Rightarrow \quad \tau = 0$$

$\Rightarrow$ no shear stress on principal planes

Visualizing Stress

1) Lamé Stress Ellipsoid

Principal frame: $[\underline{\underline{\sigma}}] = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix}$ $\qquad \{\underline{e}'_i\}$

$$\underline{\underline{\sigma}} = \sum_{i=1}^{3} \sigma_i (\underline{e}'_i \otimes \underline{e}'_i)$$

keep sum because i repeated 3 times

traction: $\underline{t} = \underline{\underline{\sigma}}\, \hat{\underline{n}}$

$$\underline{t} = \sum_{i=1}^{3} \sigma_i (\underline{e}'_i \otimes \underline{e}'_i)(n_j \underline{e}'_j)$$

$$= \sum_i \sigma_i n_j \, (\underline{e}'_i \otimes \underline{e}'_i) \underline{e}'_j$$

$$\underbrace{(\underline{e}_i \cdot \underline{e}_j)}_{\delta_{ij}} \underline{e}_i$$

$$t_i \underline{e}_i = \sum_{i=1}^{3} \underbrace{(\sigma_i n_i)}_{a_i} \underline{e}_i = a_i \underline{e}_i$$

by comparison: $\boxed{t_i = \sigma_i n_i}$ (no sum)

i is a free index !

3 equation: $t_1 = \sigma_1 n_1 \quad t_2 = \sigma_2 n_2 \quad t_3 = \sigma_3 n_3$

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} \sigma_1 \\ \\ \\ \end{bmatrix} n_1 + \begin{bmatrix} \\ \sigma_2 \\ \\ \end{bmatrix} n_2 + \begin{bmatrix} \\ \\ \sigma_3 \end{bmatrix} n_3$$

$$\Rightarrow \quad n_1 = \frac{t_1}{\sigma_1} \quad n_2 = \frac{t_2}{\sigma_2} \quad n_3 = \frac{t_3}{\sigma_3}$$

Since $\hat{\underline{n}}$ is unit vector: $n_1^2 + n_2^2 + n_3^2 = 1$

substituting:

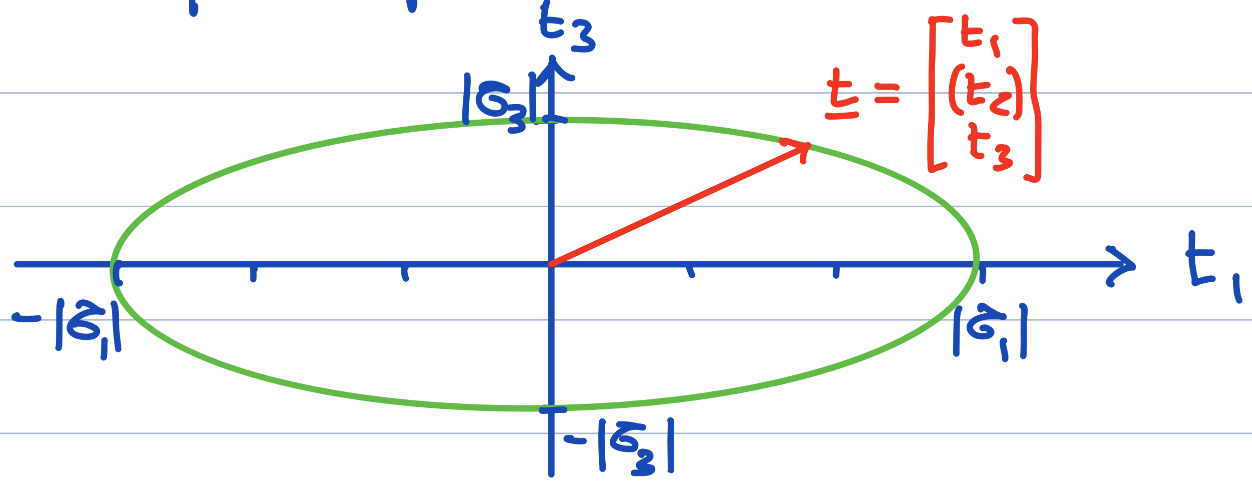
$$\boxed{\left(\frac{t_1}{\sigma_1}\right)^2 + \left(\frac{t_2}{\sigma_2}\right)^2 + \left(\frac{t_3}{\sigma_3}\right)^2 = 1}$$

Ellipse: $$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

a, b, c are semi-axes

$\Rightarrow$ Stress ellipsoid in traction space with principal stresses as semi-axes

2D

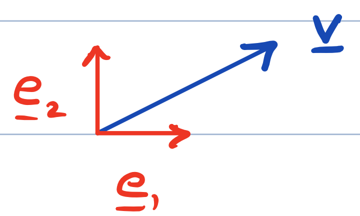


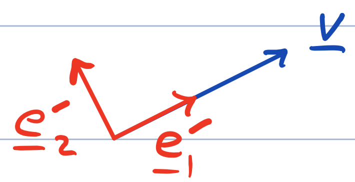
Represents the magnitude of traction vector relative to principal stresses.
$\Rightarrow$ don't know the corresponding surface!

Invariants of the stress

independent of frame $\{\underline{e}_i\}$ $\Rightarrow$ constitutive models

Example:





$[\underline{v}] = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\qquad$ $[\underline{v}]' = \begin{bmatrix} \sqrt{5} \\ 0 \end{bmatrix}$

$|\underline{v}| = \sqrt{5}$ invariant (geometric prop.)

Stress ellipse is similar geometric representation geometry ellipsoid $\Rightarrow$ invariant

3 stress invariants:

$$\boxed{\begin{aligned} &1)\ I_1 = \sigma_1 + \sigma_2 + \sigma_3 = tr(\underline{\underline{\sigma}}) \\ &2)\ I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3 = \tfrac{1}{2}\left[tr(\underline{\underline{\sigma}})^2 - tr(\underline{\underline{\sigma}}^2)\right] \\ &3)\ I_3 = \sigma_1\sigma_2\sigma_3 = \det(\underline{\underline{\sigma}}) \end{aligned}}$$

$I_3 \rightarrow$ ellipsoid volume

$V = \frac{4}{3}\pi\, a\, b\, c$ $\qquad$ a, b, c = semi-axies

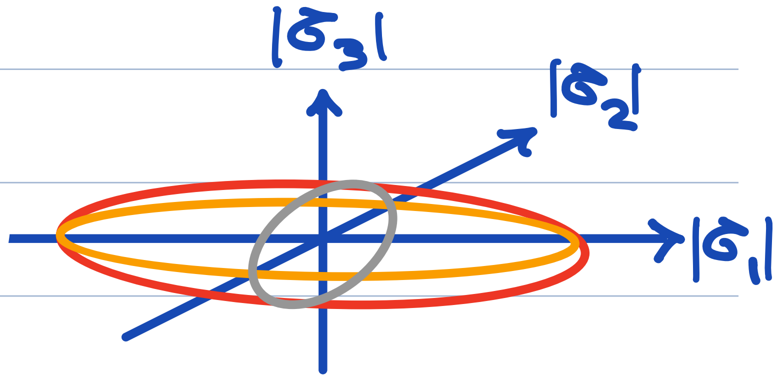
$V = \frac{4}{3}\pi\, \sigma_1 \sigma_2 \sigma_3 = \frac{4}{3}\pi I_3 \quad \Rightarrow \quad \boxed{I_3 = \frac{3V}{4\pi}}$

$I_2 \rightarrow$ sum of principal x-sectional areas

$A_{12} = \pi\,|\sigma_1 \sigma_2|$

$A_{23} = \pi\,|\sigma_2 \sigma_3|$

$A_{13} = \pi\,|\sigma_1 \sigma_3|$



$$A_{sum} = \pi\left(|\sigma_1\sigma_2| + |\sigma_2\sigma_3| + |\sigma_1\sigma_3|\right)$$

$= \pi I_2$ $\quad$ if σ_i's share sign

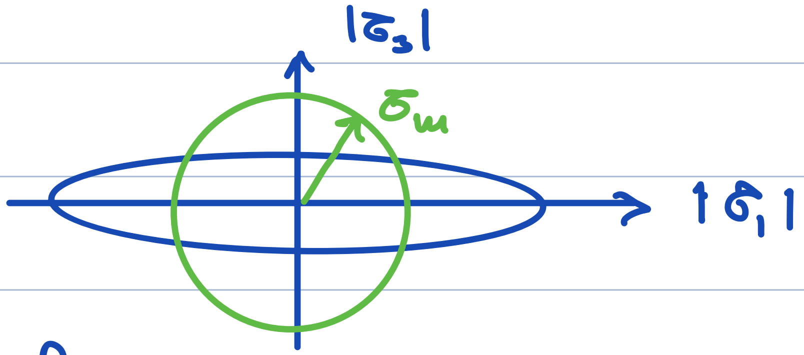
(triaxial compression/tension)

$$\boxed{I_2 = \frac{A_{sum}}{\pi}}$$

$I_1 \longrightarrow$ mean caliper diameter

mean stress: $\sigma_m = \frac{1}{3} \sum_{i=1}^{3} \sigma_i$ (hydrostatic stress)

$\boxed{I_1 = 3\,\sigma_m}$



$\Rightarrow$ σ_m is radius of
sphere with mean radius of ellipse

$\Rightarrow$ σ_m is average traction