Rate of Deformation Tensors

⇒ essential in fluid mechanics

Velocity Gradients

1) Material velocity gradient

$$\underline{\underline{F}} = \nabla_X \varphi \qquad \underline{V} = \dot{\varphi}$$

$$\dot{\underline{\underline{F}}} = \frac{\partial}{\partial t}(\nabla_X \varphi) = \nabla_X\left(\frac{\partial}{\partial t}\varphi\right) = \nabla_X \underline{V}$$

$$\Rightarrow \quad \boxed{\nabla_X \underline{V} = \dot{\underline{\underline{F}}}}$$

Note: $\varphi(\underline{X}+\underline{\Delta X},t) \approx \varphi(\underline{X},t) + \underline{\underline{F}}(\underline{X},t)\underline{\Delta X}$ $\qquad \underline{\underline{F}} = \nabla\varphi$

take material derivative

$$\dot{\varphi}(\underline{X}+\underline{\Delta X},t) \approx \dot{\varphi}(\underline{X},t) + \dot{\underline{\underline{F}}}(\underline{X},t)\,\underline{\Delta X}$$

$$\Rightarrow \dot{\varphi} = \underline{V} \qquad \dot{\underline{\underline{F}}} = \nabla \underline{V}$$

$$\boxed{\underline{V}(\underline{X}+\underline{\Delta X},t) \approx \underline{V}(\underline{X},t) + \dot{\underline{\underline{F}}}(\underline{X},t)\,\underline{\Delta X}}$$

$\Rightarrow \dot{\underline{\underline{F}}} = \nabla_X \underline{V}$ has role of $\underline{\underline{F}}$ for rates

2) Spatial velocity gradient

typically we have $\underline{v} = \underline{v}(\underline{x}, t)$

$$\underline{\underline{L}} = \nabla_x \underline{v} \qquad L_{ij} = \frac{\partial v_i}{\partial x_j}$$

Note: $\underline{V}(\underline{X}, t) = \underline{v}(\underline{\varphi}(\underline{X}, t), t)$

$\Rightarrow$ same vector field just expressed in terms of $\underline{X}$ or $\underline{x}$

$$\nabla_X \underline{V} \neq \nabla_x \underline{v}\big|_{x = \varphi(X,t)}$$

because derivatives are in different directions

$\Rightarrow$ What is relation between $\nabla_x \underline{v}$ and $\nabla_X \underline{V}$?

$$\underline{\underline{F}}^{\cdot} = \nabla_X \underline{V} \qquad \underline{V}(\underline{X}, t) = \underline{v}(\underline{\varphi}(\underline{X}, t), t)$$

$$= \nabla_X \underline{v}(\underline{\varphi}(\underline{X}, t), t)$$

$$= \nabla_X \underline{v}(\underline{x}, t) = \frac{\partial v_i}{\partial X_j} \underline{e}_i \otimes \underline{e}_j$$

$$= \frac{\partial}{\partial x_k} \frac{\partial x_k}{\partial X_j} v_i \, \underline{e}_i \otimes \underline{e}_j = \frac{\partial v_i}{\partial x_k} \frac{\partial x_k}{\partial X_j} \underline{e}_i \otimes \underline{e}_j$$

$$x_k = \varphi_k(X,t) \Rightarrow \frac{\partial x_k}{\partial X_j} = \frac{\partial \varphi_k}{\partial X_j} = F_{kj}$$

$$\Rightarrow \dot{\underline{\underline{F}}} = v_{i,k} F_{kj} \underline{e}_i \otimes \underline{e}_j = \nabla_x \underline{v}\, \underline{\underline{F}}$$

$$\Rightarrow \boxed{\begin{aligned} \underline{\underline{L}} &= \nabla_x \underline{v} = \dot{\underline{\underline{F}}}\, \underline{\underline{F}}^{-1} = \nabla_x \underline{V}\, \underline{\underline{F}}^{-1} \\ \dot{\underline{\underline{F}}} &= \nabla_X \underline{V} = \underline{\underline{L}}\, \underline{\underline{F}} = \nabla_x \underline{v}\, \underline{\underline{F}} \end{aligned}}$$

$\Rightarrow$ important in constitutive theory

Symmetric-Skew decomposition of $\underline{\underline{L}}$

$$\nabla_x \underline{v} \equiv \nabla \underline{v}$$

$$\boxed{\begin{aligned} \underline{\underline{L}} &= \underline{\underline{D}} + \underline{\underline{W}} \\ \underline{\underline{D}} &= \operatorname{sym}(\nabla \underline{v}) = \tfrac{1}{2}(\nabla \underline{v} + \nabla \underline{v}^T) \\ \underline{\underline{W}} &= \operatorname{skew}(\nabla \underline{v}) = \tfrac{1}{2}(\nabla \underline{v} - \nabla \underline{v}^T) \end{aligned}}$$

$\underline{\underline{D}}$ rate of strain tensor

$\underline{\underline{W}}$ spin tensor

$\underline{\underline{D}}$ = rate of change in shape (strech rate)

$\underline{\underline{W}}$ = rate of change in orientation (spin)

Note: Definition of the rate of strain tensor

$$\underline{\underline{D}} = \frac{1}{2}(\nabla \underline{v} + \nabla \underline{v})$$

is unique but the definition of strain tensor is not!

Streching rate of line element

$$\underline{dx} = \ell \, \underline{n} \qquad n = \frac{\underline{dx}}{|\underline{dx}|} \qquad \underline{n} \cdot \underline{n} = 1$$

$$\underline{dx} = \underline{\underline{F}} \, \underline{dX}$$

$$\dot{\underline{dx}} = \dot{\underline{\underline{F}}} \, \underline{dX} = \underline{\underline{L}} \, \underbrace{\underline{\underline{F}} \, \underline{dX}}_{\underline{dx}} = \underline{\underline{L}} \, \underline{dx}$$

$$\Rightarrow \dot{\underline{dx}} = \underline{\underline{L}} \, \underline{dx}$$

$$\ell^2 = \underline{dx} \cdot \underline{dx}$$

$$\text{l.h.s.: } \dot{(\ell^2)} = 2 \ell \dot{\ell}$$

$$\text{r.h.s.: } \dot{(\underline{dx} \cdot \underline{dx})} = \dot{\underline{dx}} \cdot \underline{dx} + \underline{dx} \cdot \dot{\underline{dx}}$$

$$= 2 \, \underline{dx} \cdot \dot{\underline{dx}}$$

$$\Rightarrow \quad \ell\dot{\ell} = \underline{dx} \cdot \underline{\dot{dx}} \qquad \underline{\dot{dx}} = \underline{\underline{L}}\,\underline{dx}$$

$$= \underline{dx} \cdot \underline{\underline{L}}\,\underline{dx} \qquad \underline{dx} = \ell\,\underline{n}$$

$$\ell\dot{\ell} = \ell^2\, \underline{n} \cdot \underline{\underline{L}}\,\underline{n}$$

$$\Rightarrow \quad \frac{\dot{\ell}}{\ell} = \underline{n} \cdot \underline{\underline{L}}\,\underline{n} \qquad \underline{\underline{L}} = \underline{\underline{D}} + \underline{\underline{W}}$$

↑ spin?

$$\underline{n} \cdot \underline{\underline{L}}\,\underline{n} = \underline{n} \cdot (\underline{\underline{D}} + \underline{\underline{W}})\,\underline{n}$$

$$= \underline{n} \cdot \underline{\underline{D}}\,\underline{n} + \underline{n} \cdot \underline{\underline{W}}\,\underline{n}^{\,0} \qquad \underline{\underline{W}} = -\underline{\underline{W}}^T$$

$$= \underline{n} \cdot \underline{\underline{D}}\,\underline{n}$$

$$\underline{n} \cdot \underline{\underline{W}}\,\underline{n} = n_i W_{ij} n_j = W_{ij} n_i n_j = -W_{ji} n_i n_j$$

$$= -W_{ij} n_i n_j = 0$$

$$\Rightarrow \quad \boxed{\frac{\dot{\ell}}{\ell} = \underline{n} \cdot \underline{\underline{D}}\,\underline{n}}$$

How fast are two material points separating right now?

Both ℓ and $\dot{\ell}$ are at current time

so no reference state is needed !

$\Rightarrow$ unique

Strain asks how much has it deformed relative to some previous state ℓ_0.

How do you measure this?

- ℓ / ℓ_0 = stretch
- $(\ell - \ell_0)/\ell_0$ = engineering strain
- $(\ell - \ell_0]/\ell$
- $\ln(\ell/\ell_0)$ = natural / Hencky strain

Speed is unambiguous but distance traveled depends on the route and how you measure it.

Which strain tensor connects to $\underline{\underline{D}}$?

Connection to stretches

$$\lambda(\underline{N},t) = \frac{\ell}{\ell_0} = \sqrt{\underline{N}\cdot\underline{\underline{C}}\,\underline{N}} \qquad \ell = \lambda \ell_0$$

$$\underline{dx} = \underline{\underline{F}}\,\underline{dX} \qquad \ell = |\underline{dx}| \quad \ell_0 = |\underline{dX}|$$

$$\ell\,\underline{n} = \underline{\underline{F}}\,\ell_0\,\underline{N}$$

$$\frac{\ell}{\ell_0}\underline{n} = \underline{\underline{F}}\,\underline{N} \quad \Rightarrow \quad \underline{\underline{F}}\,\underline{N} = \lambda\,\underline{n}$$

$$\lambda^2 = \underline{N} \cdot \underline{\underline{C}}\, \underline{N}$$

differentiate ($\underline{N}$ = const)

$$2\lambda\dot{\lambda} = \underline{N} \cdot \dot{\underline{\underline{C}}}\, \underline{N} \qquad \dot{\underline{\underline{C}}} = 2\dot{\underline{\underline{E}}} = 2\underline{\underline{F}}^T \dot{\underline{\underline{D}}}\, \underline{\underline{F}}$$

$$\lambda\dot{\lambda} = \underline{N} \cdot \underline{\underline{F}}^T \dot{\underline{\underline{D}}}\, \underline{\underline{F}}\, \underline{N} = \underline{\underline{F}}\underline{N} \cdot \dot{\underline{\underline{D}}}\, \underline{\underline{F}}\underline{N} = \lambda^2\, \underline{n} \cdot \dot{\underline{\underline{D}}}\, \underline{n}$$

$$\Rightarrow \qquad \boxed{\frac{\dot{\lambda}}{\lambda} = \underline{n} \cdot \dot{\underline{\underline{D}}}\, \underline{n}}$$

Check consistency with previous result

$$\underline{n} \cdot \dot{\underline{\underline{D}}}\, \underline{n} = \frac{\dot{\ell}}{\ell} = \frac{\dot{(\lambda \ell_0)}}{\ell} = \frac{\dot{\lambda}\,\ell_0}{\ell} = \frac{\dot{\lambda}}{\lambda}$$

$\underline{\underline{D}}$ connects naturally to Hencky strain

$$\boxed{\frac{\dot{\lambda}}{\lambda} = \frac{d}{dt} \ln(\lambda)}$$

Relation between Strain & Strain Rate

Green-Lagrange strain tensor

$$\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}}) = \frac{1}{2}(\underline{\underline{F}}^T\underline{\underline{F}} - \underline{\underline{I}})$$

Strain rate $\dot{\underline{\underline{E}}}$

$$\dot{\underline{\underline{E}}} = \frac{1}{2}(\dot{\underline{\underline{F}}}^T\underline{\underline{F}} + \underline{\underline{F}}^T\dot{\underline{\underline{F}}}) \qquad \text{where } \dot{\underline{\underline{F}}} = \underline{\underline{\ell}}\,\underline{\underline{F}}$$

$$= \frac{1}{2}\left[(\underline{\underline{\ell}}\,\underline{\underline{F}})^T\underline{\underline{F}} + \underline{\underline{F}}^T\underline{\underline{\ell}}\,\underline{\underline{F}}\right]$$

$$= \frac{1}{2}(\underline{\underline{F}}^T\underline{\underline{\ell}}^T\underline{\underline{F}} + \underline{\underline{F}}^T\underline{\underline{\ell}}\,\underline{\underline{F}})$$

$$= \underline{\underline{F}}^T\frac{1}{2}(\underline{\underline{\ell}}^T + \underline{\underline{\ell}})\underline{\underline{F}} = \underline{\underline{F}}^T\underline{\underline{d}}\,\underline{\underline{F}}$$

$$\boxed{\dot{\underline{\underline{E}}} = \underline{\underline{F}}^T\underline{\underline{d}}\,\underline{\underline{F}}}$$

Small deformation: (Lecture 16)

$$\underline{\underline{E}} \approx \underline{\underline{\varepsilon}} = \frac{1}{2}(\nabla\underline{u} + \nabla\underline{u}^T)$$

$$\underline{\underline{F}} \approx \underline{\underline{I}} + \nabla\underline{u}$$

$$\dot{\underline{\underline{\varepsilon}}} \approx \dot{\underline{\underline{E}}} \approx (\underline{\underline{I}} + \nabla\underline{u}^T)\,\underline{\underline{d}}\,(\underline{\underline{I}} + \nabla\underline{u})$$

$$= \underline{\underline{d}} + \underbrace{\nabla\underline{u}^T\underline{\underline{d}}}_{\delta^2} + \underbrace{\nabla\underline{u}^T\underline{\underline{d}}\,\nabla\underline{u}}_{\delta^3}$$

for small deformation

$$\dot{\underline{\underline{\varepsilon}}} = \underline{\underline{D}} = \frac{1}{2}\left(\nabla \underline{v} + \nabla \underline{v}^T\right)$$

What about larger strains?

start $\quad \dfrac{\dot{\lambda}}{\lambda} = \underline{n} \cdot \underline{\underline{D}}\, \underline{n} \qquad \dfrac{\dot{\lambda}}{\lambda} = \dfrac{d}{dt} \ln(\lambda)$

obtain λ from $\dot{\lambda}(\underline{\underline{D}})$ by integration

$$\Rightarrow \quad \frac{d}{dt} \ln(\lambda) = \underline{n} \cdot \underline{\underline{D}}\, \underline{n}$$

$$\ln(\lambda) = \int_0^t \underline{n} \cdot \underline{\underline{D}}\, \underline{n}\, dt \qquad \lambda(t=0) = 1$$

Logarithmic strain is sum of incremental strains

$$\lambda(\underline{N}, t) = \sqrt{\underline{N} \cdot \underline{\underline{C}}\, \underline{N}}$$

$$\ln \lambda(\underline{N}, t) = \frac{1}{2} \ln\left(\underline{N} \cdot \underline{\underline{C}} N\right)$$

we would like: $\quad \ln\left(\underline{N} \cdot \underline{\underline{C}}\, \underline{N}\right) = \underline{N} \cdot \ln\left(\underline{\underline{C}}\right) \underline{N}$

only true if $\underline{N}$ is principal direction

Assume fixed principal directions $\underline{N}_i = \underline{n}_i$

(typically the case in rock mechanics experiments)

$$\lambda(\underline{N}_i, t) = \lambda_i$$

$$\ln \lambda_i = \frac{1}{2} \ln \left(\underline{N}_i \cdot \underline{\underline{C}}\, \underline{N}_i \right) =$$

$$= \frac{1}{2}\, \underline{N}_i \cdot \ln(\underline{\underline{C}})\, \underline{N}_i \qquad \underline{\underline{C}} = \underline{\underline{U}}^2$$

$$= \underline{N}_i \circ \underbrace{\ln(\underline{\underline{U}})}_{\underline{\underline{H}}}\, \underline{N}_i$$

Hencky strain:

$$\boxed{\underline{\underline{H}} = \ln(\underline{\underline{U}}) = \sum_{i=1}^{3} \ln(\lambda_i)\, \underline{N}_i \otimes \underline{N}_i}$$

3D version $\ln\left(\frac{\ell}{\ell_0}\right)$

If there is no rotation $\underline{\underline{R}} = \underline{\underline{I}} \Rightarrow \underline{\underline{F}} = \underline{\underline{U}} = \underline{\underline{V}}$

$$\Rightarrow \boxed{\overset{\circ}{\underline{\underline{H}}} = \underline{\underline{D}}}$$

Note: General rotating case is more complex!