Tensor scalar product (Contraction)

analogous to scalar product of vectors

$$\boxed{\underline{\underline{A}} : \underline{\underline{B}} = tr(\underline{\underline{A}}^T \underline{\underline{B}}) = A_{ij} B_{ij}}$$

scalar !

explicitly:

$$\underline{\underline{A}} : \underline{\underline{B}} = \sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij} B_{ij} = A_{11} B_{11} + A_{12} B_{12} + A_{13} B_{13} + \dots$$
$$A_{21} B_{21} + A_{22} B_{22} + A_{23} B_{23} + \dots$$
$$A_{31} B_{31} + A_{32} B_{32} + A_{33} B_{33}$$

The index expression is derived as follows

$$\underline{\underline{A}} : \underline{\underline{B}} = tr(\underline{\underline{A}}^T \underline{\underline{B}}) =$$

$$\underline{\underline{A}}^T \underline{\underline{B}} = (A_{ji} \underline{e}_i \otimes \underline{e}_j)(B_{kl} \underline{e}_k \otimes \underline{e}_l)$$
$$= A_{ji} B_{kl} (\underline{e}_i \otimes \underline{e}_j)(\underline{e}_k \otimes \underline{e}_l)$$
$$= A_{ji} B_{kl} (\underline{e}_j \cdot \underline{e}_k)(\underline{e}_i \otimes \underline{e}_l) = A_{ji} B_{kl} \delta_{jk} \, \underline{e}_i \otimes \underline{e}_l$$
$$= A_{ji} B_{jl} \, \underline{e}_i \otimes \underline{e}_l$$

$$tr(\underline{\underline{A}}^T \underline{\underline{B}}) = A_{ji} B_{jl} \delta_{il} = A_{ji} B_{ji} = A_{ij} B_{ij} \quad \checkmark$$

Properties: 1) $\underline{\underline{A}} : \underline{\underline{B}} = \underline{\underline{B}} : \underline{\underline{A}}$

2) $(\underline{a} \otimes \underline{b}) : (\underline{c} \otimes \underline{d}) = (\underline{a} \cdot \underline{c})(\underline{b} \cdot \underline{d})$

First follows from prop. of trace

$$\underline{\underline{A}} : \underline{\underline{B}} = tr(\underline{\underline{A}}^T \underline{\underline{B}}) = tr((\underline{\underline{A}}^T \underline{\underline{B}})^T) = tr(\underline{\underline{B}}^T \underline{\underline{A}}) = \underline{\underline{B}} : \underline{\underline{A}}$$

Second property $[\underline{a} \otimes \underline{b}]_{ij} = a_i b_j$

$$(\underline{a} \otimes b) : (\underline{c} \otimes \underline{d}) = [\underline{a} \otimes \underline{b}]_{ij} [\underline{c} \otimes \underline{d}]_{ij} = a_i b_j c_i d_j$$
$$= a_i c_i b_j d_j$$
$$= (\underline{a} \cdot \underline{c})(\underline{b} \cdot \underline{d})$$

A common norm for tensors is

$$\boxed{|\underline{\underline{A}}| = \sqrt{\underline{\underline{A}} : \underline{\underline{A}}} = \sqrt{A_{ij} A_{ij}} \geq 0}$$

Note: Tensor scalar product will be important to express the work done during deformation. For example the shear heating in glaciology.

add sym skew contraction $\Rightarrow$ useful

also important for deriv. of tensor functions for $\nabla \times \nabla \phi = 0$