

Why do we need tensors?

Scalars:

describe a quantity at a point

e.g. Temperature

Vectors:

describe quantity and a direction

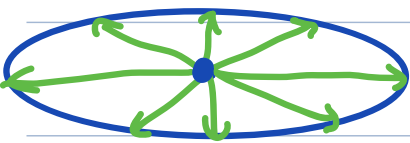
e.g. velocity (speed + direction)



Tensors:

describes how a quantity changes with direction

Think of an ellipsoid



Examples: anisotropic properties

stress, strain

moment of inertia

Second-order Tensors

Linear operators : $\underline{v} = \underline{A} \underline{u}$

maps vector $\underline{u} \in \mathcal{V}$ into vector $\underline{v} \in \mathcal{V}$

$\underline{A}$ and $\underline{B}$ are equal: $\underline{A} \underline{v} = \underline{B} \underline{v}$ for all $\underline{v} \in \mathcal{V}$

Special Tensors

Zero tensor: $\underline{0} \underline{v} = \underline{0}$ for all $\underline{v} \in \mathcal{V}$

Identity tensor: $\underline{I} \underline{v} = \underline{v}$ for all $\underline{v} \in \mathcal{V}$

Basic algebraic operations

α = scalar, $\underline{v}$ = vector, $\underline{A}$ & $\underline{B}$ 2nd-ord. tensors

1) $(\alpha \underline{A}) \underline{v} = \underline{A} (\alpha \underline{v})$ scalar multiplication

2) $(\underline{A} + \underline{B}) \underline{v} = \underline{A} \underline{v} + \underline{B} \underline{v}$ tensor sum

3) $(\underline{A} \underline{B}) \underline{v} = \underline{A} (\underline{B} \underline{v})$ tensor product

4) (tensor scalar product $\rightarrow$ later)

1 + 2 $\Rightarrow$ imply linearity

1, 2, 3 produce other tensors

set $\mathcal{V}^2$ of second order tensors $\Rightarrow$ vector space

Q: What is a basis for $\mathcal{V}^2$?

Representation of a tensor

In a frame $\{\underline{e}_i\}$ a second order tensor $\underline{\underline{S}}$ is represented by nine numbers

$$S_{ij} = \underline{e}_i \cdot \underline{\underline{S}} \underline{e}_j$$

Matrix representation of tensor in $\{\underline{e}_i\}$

$$[\underline{\underline{S}}] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \in \mathbb{R}^3 \times \mathbb{R}^3$$

Consider $\underline{v} = \underline{\underline{S}} \underline{u}$ where $\underline{v} = v_k \underline{e}_k$, $\underline{u} = u_j \underline{e}_j$

$$v_k \underline{e}_k = \underline{\underline{S}} (u_j \underline{e}_j) = \underline{\underline{S}} \underline{e}_j u_j$$

multiply by $\underline{e}_i$ from left

$$v_k \underline{e}_i \cdot \underline{e}_k = \underline{e}_i \cdot \underline{\underline{S}} \underline{e}_j u_j$$

$$v_k \delta_{ik} = \underline{e}_i \cdot \underline{\underline{S}} \underline{e}_j u_j$$

$$v_i = (\underline{e}_i \cdot \underline{\underline{S}} \underline{e}_j) u_j$$

$$v_i = S_{ij} u_j$$

Example: Multiply tensor by vector

$$\underline{\underline{S}} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 4 & 2 \\ 3 & 2 & 6 \end{bmatrix} \quad \underline{u} = \begin{bmatrix} 7 \\ 3 \\ 6 \end{bmatrix}$$

Dot product with rows of $\underline{\underline{S}}$:

index notation: $v_i = S_{ij} u_j = \sum_{j=1}^3 S_{ij} u_j$

$$i=1: v_1 = S_{11} u_1 + S_{12} u_2 + S_{13} u_3 = 1 \cdot 7 + 0 \cdot 3 + 3 \cdot 6 = 25$$

$$\underline{\underline{S}} = \begin{bmatrix} - & \underline{\underline{S}}_1 & - \\ - & \underline{\underline{S}}_2 & - \\ - & \underline{\underline{S}}_3 & - \end{bmatrix}$$

$$v_i = \underline{\underline{S}}_i \cdot \underline{u}$$

$$\underline{v} = v_i \underline{e}_i = (\underline{\underline{S}}_i \cdot \underline{u}) \underline{e}_i$$

Linear combination of columns of $\underline{\underline{S}}$:

$$\underline{v} = \underline{\underline{S}} \underline{u} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 4 & 2 \\ 3 & 2 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 6 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} + 6 \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 0 \\ 21 \end{bmatrix} + \begin{bmatrix} 0 \\ 12 \\ 6 \end{bmatrix} + \begin{bmatrix} 18 \\ 12 \\ 36 \end{bmatrix} = \begin{bmatrix} 25 \\ 24 \\ 63 \end{bmatrix}$$

Dyadic Product

The dyadic product of two vectors $\underline{a}$ and $\underline{b}$ is the 2nd-order tensor $\underline{a} \otimes \underline{b}$ defined by

$$(\underline{a} \otimes \underline{b}) \underline{v} = (\underline{b} \cdot \underline{v}) \underline{a} \quad \text{for all } \underline{v} \in \mathcal{V}$$

This has the form: $\underline{S} \underline{v} = \alpha \underline{a}$

in components: $S_{ij} v_j = \alpha a_i$

$$\alpha = \underline{b} \cdot \underline{v} = b_j v_j$$

$$S_{ij} = [\underline{a} \otimes \underline{b}]_{ij}$$

$$\Rightarrow [\underline{a} \otimes \underline{b}]_{ij} v_j = b_j v_j a_i$$

$$[\underline{a} \otimes \underline{b}]_{ij} v_j = (a_i b_j) v_j$$

$$\Rightarrow [\underline{a} \otimes \underline{b}]_{ij} = a_i b_j$$

So that

$$[\underline{a} \otimes \underline{b}] = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} = \underline{a} \underline{b}^T$$

Linearity of dyadic product:

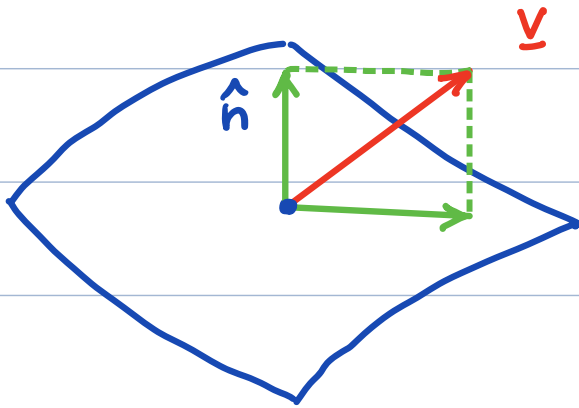
Linearity of dyadic product:

scalars α, β & vectors $\underline{a}, \underline{b}, \underline{v}, \underline{w}$

$$(\underline{a} \otimes \underline{b})(\alpha \underline{v} + \beta \underline{w}) = \alpha(\underline{a} \otimes \underline{b})\underline{v} + \beta(\underline{a} \otimes \underline{b})\underline{w}$$

Projection tensors

commonly used to partition forces on a surface.



$$\underline{v} = \underline{v}_n^{\parallel} + \underline{v}_n^{\perp}$$

Use dot product:

$$\underline{v}_n^{\parallel} = (\underline{v} \cdot \underline{\hat{n}}) \underline{\hat{n}}$$

$$\underline{v}_n^{\perp} = \underline{v} - \underline{v}_n^{\parallel}$$

$$\text{Tensors? } \underline{v}_n^{\parallel} = \underline{P}_n^{\parallel} \underline{v}, \quad \underline{v}_n^{\perp} = \underline{P}_n^{\perp} \underline{v}$$

use dyadic property!

$$\underline{v}_n^{\parallel} = (\underline{v} \cdot \underline{\hat{n}}) \underline{\hat{n}} = (\underline{\hat{n}} \otimes \underline{\hat{n}}) \underline{v} = \underline{P}_n^{\parallel} \underline{v}$$

$$\underline{v}_n^{\perp} = \underline{v} - (\underline{\hat{n}} \otimes \underline{\hat{n}}) \underline{v} = (\underline{I} - \underline{\hat{n}} \otimes \underline{\hat{n}}) \underline{v} = \underline{P}_n^{\perp} \underline{v}$$

Projection tensors:

$$\underline{P}_n^{\parallel} = \underline{\hat{n}} \otimes \underline{\hat{n}}$$

$$\underline{P}_n^{\perp} = \underline{I} - \underline{\hat{n}} \otimes \underline{\hat{n}}$$

Basis for V^2

Given frame $\{\underline{e}_i\}$ the dyadic products $\{\underline{e}_i \otimes \underline{e}_j\}$ form a basis for vector space of tensors:

$$\underline{\underline{S}} = S_{ij} \underline{e}_i \otimes \underline{e}_j \quad \text{where} \quad S_{ij} = \underline{e}_i \cdot \underline{\underline{S}} \underline{e}_j$$

Consider $\underline{v} = \underline{\underline{S}} \underline{u}$ with $\underline{v} = v_i \underline{e}_i$, $\underline{u} = u_k \underline{e}_k$

$$\begin{aligned} v_i \underline{e}_i &= S_{ij} (\underline{e}_i \otimes \underline{e}_j) (u_k \underline{e}_k) \\ &= S_{ij} u_k (\underline{e}_i \otimes \underline{e}_j) \underline{e}_k \quad \text{use dyadic property} \\ &= S_{ij} u_k (\underline{e}_j \cdot \underline{e}_k) \underline{e}_i \\ &^* = S_{ij} u_k \delta_{jk} \underline{e}_i \\ &= S_{ij} u_j \underline{e}_i \end{aligned}$$

$$v_i \underline{e}_i = S_{ij} u_j \underline{e}_i$$

by comparison: $v_i = S_{ij} u_j$ as before

Note: Transfer property of δ_{ij} applies to tensors

$$^* = \underbrace{S_{ij} \delta_{jk}}_{S_{ik}} u_k \underline{e}_i$$

$$= S_{ik} u_k \underline{e}_i$$

Dyadic property $\Rightarrow$ tensor vector products

What about tensor products?

$$\underline{\underline{S}} \underline{\underline{T}} = S_{ij} (\underline{e}_i \otimes \underline{e}_j) T_{kl} (\underline{e}_k \otimes \underline{e}_l) = S_{ij} T_{ij} (\underline{e}_i \otimes \underline{e}_j) (\underline{e}_k \otimes \underline{e}_l)$$

Product of dyadic products:

$$(\underline{a} \otimes \underline{b})(\underline{c} \otimes \underline{d}) = (\underline{b} \cdot \underline{c}) \underline{a} \otimes \underline{d} \Rightarrow \text{HW2}$$

$$\begin{aligned} \underline{\underline{H}} = \underline{\underline{S}} \underline{\underline{T}} &= S_{ij} T_{kl} (\underline{e}_i \otimes \underline{e}_j) (\underline{e}_k \otimes \underline{e}_l) \\ &= S_{ij} T_{kl} \underbrace{(\underline{e}_j \cdot \underline{e}_k)}_{\delta_{jk}} \underline{e}_i \otimes \underline{e}_l \end{aligned}$$

$$= S_{ik} T_{kl} \underline{e}_i \otimes \underline{e}_l \quad \text{rename } l \leftrightarrow j$$

$$H_{ij} \underline{e}_i \otimes \underline{e}_j = S_{ik} T_{kj} \underline{e}_i \otimes \underline{e}_j$$

$$\Rightarrow \boxed{H_{ij} = S_{ik} T_{kj}} \quad \text{note the dummy } k!$$

Transpose of a Tensor

Every tensor $\underline{\underline{S}}$ has a transpose $\underline{\underline{S}}^T$ so that

$$\underline{\underline{S}} \underline{u} \cdot \underline{v} = \underline{u} \cdot \underline{\underline{S}}^T \underline{v} \quad \text{for all } \underline{u}, \underline{v} \in \mathcal{V}$$

Implies $S_{ij}^T = S_{ji}$:

$$(S_{ij} u_j e_i) \cdot (v_l e_l) = (u_k e_k) \cdot (S_{ij}^T v_j e_i)$$

$$S_{ij} u_j v_l (\underbrace{e_i \cdot e_l}_{\delta_{il}}) = S_{ij}^T u_k v_j (\underbrace{e_k \cdot e_i}_{\delta_{ki}})$$

$$S_{ij} u_j v_i = S_{ij}^T u_i v_j \quad \text{indices on } u \& v$$

$$S_{ji} u_i v_j = S_{ij}^T u_i v_j \quad i \leftrightarrow j \text{ on l.h.s}$$

$$\Rightarrow \underline{\underline{S}}_{ij}^T = \underline{\underline{S}}_{ji} \quad \text{index notation for transpose}$$

Properties: 1) $(\underline{\underline{A}}^T)^T = \underline{\underline{A}}$

2) $(\underline{\underline{A}} \underline{\underline{B}})^T = \underline{\underline{B}}^T \underline{\underline{A}}^T$

3) $(\underline{u} \otimes \underline{v})^T = \underline{v} \otimes \underline{u}$

Note: Cannot transpose vector, only one index!

Symmetric & Skew Tensors

$\underline{\underline{S}}$ is symmetric if $\underline{\underline{S}} = \underline{\underline{S}}^T$

$$S_{ij} = S_{ji}$$

$\underline{\underline{S}}$ is skew if $\underline{\underline{S}} = -\underline{\underline{S}}^T$

$$S_{ij} = -S_{ji}$$

skew tensors related to rotation

Symmetric - Skew Decomposition

$$\underline{\underline{S}} = \underline{\underline{E}} + \underline{\underline{W}}$$

$$\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{S}} + \underline{\underline{S}}^T)$$

$$\underline{\underline{W}} = \frac{1}{2}(\underline{\underline{S}} - \underline{\underline{S}}^T)$$

$$\underline{\underline{E}} = \underline{\underline{E}}^T$$

$$\underline{\underline{W}} = -\underline{\underline{W}}^T$$

Trace of Tensor

$$\text{tr}(\underline{\underline{S}}) = S_{ii} = S_{11} + S_{22} + S_{33}$$

Properties: 1) $\text{tr}(\underline{\underline{A}}^T) = \text{tr}(\underline{\underline{A}})$

2) $\text{tr}(\underline{\underline{A}} \underline{\underline{B}}) = \text{tr}(\underline{\underline{B}} \underline{\underline{A}})$

3) $\text{tr}(\underline{\underline{A}} + \underline{\underline{B}}) = \text{tr}(\underline{\underline{A}}) + \text{tr}(\underline{\underline{B}})$

4) $\text{tr}(\alpha \underline{\underline{A}}) = \alpha \text{tr}(\underline{\underline{A}})$

Trace of dyadic product: $\text{tr}(\underline{a} \otimes \underline{b}) = a_i b_i$

Prop. 4

$$\begin{aligned}\text{tr}(\underline{S}) &= \text{tr}(S_{ij} \underline{e}_i \otimes \underline{e}_j) \stackrel{\text{Prop. 4}}{=} S_{ij} \underbrace{\text{tr}(\underline{e}_i \otimes \underline{e}_j)}_{\delta_{ij}} \\ &= S_{ij} \delta_{ij} \\ &= S_{ii}\end{aligned}$$

Spherical - Deviatoric Decomposition

$$\underline{S} = \alpha \underline{I} + \text{dev}(\underline{S})$$

Spherical tensor: $\text{sph}(\underline{S}) = \alpha \underline{I}$ where $\alpha = \frac{1}{3} \text{tr}(\underline{S})$

Deviatoric tensor: $\text{dev}(\underline{S}) = \underline{S} - \alpha \underline{I}$

Important

Spherical part $\rightarrow$ volumetric stress/strain

Deviatoric part $\rightarrow$ shear stress/strain

Note: $\text{tr}(\text{dev}(\underline{S})) = 0$ by design

Determinant and Inverse

$$\det(\underline{\underline{A}}) = \det \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \epsilon_{ijk} [\underline{\underline{A}}]_{i1} [\underline{\underline{A}}]_{j2} [\underline{\underline{A}}]_{k3}$$

where $[\underline{\underline{A}}]_{i1}$, $[\underline{\underline{A}}]_{j2}$, $[\underline{\underline{A}}]_{k3}$ are the columns of $[\underline{\underline{A}}]$

Triple scalar product: $(\underline{a} \times \underline{b}) \cdot \underline{c} = \det[\underline{a}, \underline{b}, \underline{c}] = \epsilon_{ijk} a_i b_j c_k$
determinants $\Rightarrow$ volumes

Properties:

- 1) $\det(\underline{\underline{A}} \underline{\underline{B}}) = \det(\underline{\underline{A}}) \det(\underline{\underline{B}})$
- 2) $\det(\underline{\underline{A}}^T) = \det(\underline{\underline{A}})$
- 3) $\det(\alpha \underline{\underline{A}}) = \alpha^n \det(\underline{\underline{A}})$ ($\underline{\underline{A}}$ is $n \times n$)

$\underline{\underline{A}}$ is singular if $\det \underline{\underline{A}} = 0$.

If $\det \underline{\underline{A}} \neq 0$ then the inverse $\underline{\underline{A}}^{-1}$ exists

$$\underline{\underline{A}}^{-1} \underline{\underline{A}} = \underline{\underline{A}} \underline{\underline{A}}^{-1} = \underline{\underline{I}}$$