

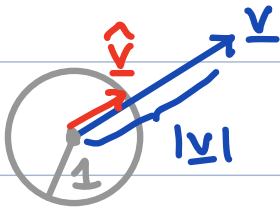
Vectors and index notation

Def: Vector $\underline{v}$ is a quantity with a magnitude and a direction

$$\underline{v} = |\underline{v}| \hat{\underline{v}}$$

$|\underline{v}| \geq 0$ magnitude (scalar)

$\hat{\underline{v}} = \frac{\underline{v}}{|\underline{v}|}$ direction (unit vector)



Physical examples: velocity, force, heat flux

Q: Is it possible to have a vector without direction?

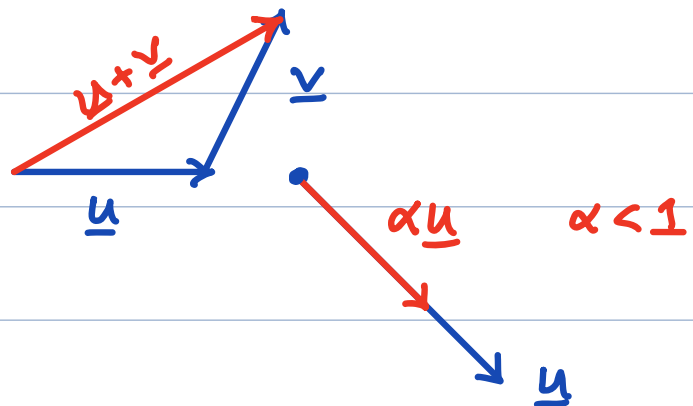
Def: Vector space, $\mathcal{V}$, is a collection of objects that is closed under addition and scalar multiplication

$$\underline{u} \in \mathcal{V} \quad \underline{v} \in \mathcal{V}$$

$$\alpha \in \mathbb{R}$$

$$1) \quad \underline{u} + \underline{v} \in \mathcal{V}$$

$$2) \quad \alpha \underline{u} \in \mathcal{V}$$



Q: Do vectors in $\mathbb{R}^3$ form a vector space?

Basis for a vector space

Def.: Basis for $\mathcal{V}$ is a set of linearly independent vectors $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ that span the space (3D).

components of vector in basis

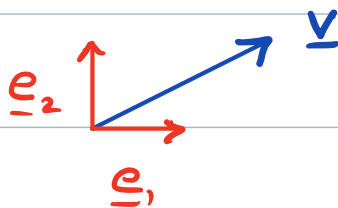
$$\underline{v} = v_1 \underline{e}_1 + v_2 \underline{e}_2 + v_3 \underline{e}_3 \quad [\underline{v}] = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Here $[\underline{v}]$ is the representation of $\underline{v}$ in $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

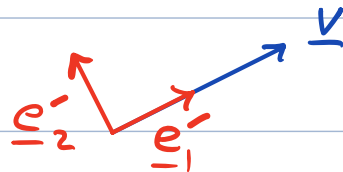
vector $\longleftrightarrow$ representation

basis is not unique !

Example:



$$[\underline{v}] = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



$$[\underline{v}]' = \begin{bmatrix} \sqrt{5} \\ 0 \end{bmatrix}$$

same vector $\underline{v}$ two different representations !

Reference frame

any orthonormal basis $\{\underline{e}\} = \{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

normal: $|\underline{e}_1| = |\underline{e}_2| = |\underline{e}_3| = 1$

ortho: $\underline{e}_1 \perp \underline{e}_2, \underline{e}_1 \perp \underline{e}_3, \underline{e}_2 \perp \underline{e}_3$

Q: Any additional common restrictions on basis?

Index notation

1) Dummy index

$$\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$$

$$\underline{v} = v_1 \underline{e}_1 + v_2 \underline{e}_2 + v_3 \underline{e}_3 = \sum_{i=1}^3 v_i \underline{e}_i \equiv v_i \underline{e}_i$$

If index is repeated twice in a term

$\Rightarrow$ summation is implied

(Einstein summation convention)

$\Rightarrow$ dummy index

Note: $\underline{a} = a_i \underline{e}_i = a_k \underline{e}_k = a_q \underline{e}_q$

$\Rightarrow$ rename dummy indices

Example: Collecting terms

$$\underline{a} = a_i \underline{e}_i \quad \underline{b} = b_j \underline{e}_j$$

$$\begin{aligned} \underline{a} + \underline{b} &= a_i \underline{e}_i + b_j \underline{e}_j && \text{rename } j \rightarrow i \\ &= a_i \underline{e}_i + b_i \underline{e}_i \end{aligned}$$

$$= (a_i + b_i) \underline{e}_i$$

2) Free index

occurs only once in a term

Example: $a_i = c_j b_j b_i$ $j = \text{dummy index}$
 $i = \text{free index}$

$\Rightarrow$ set of equations $i, j \in \{1, 2, 3\}$

$$a_i = \left(\sum_{j=1}^3 c_j b_j \right) b_i, \quad a_2 = \left(\sum_{j=1}^3 c_j b_j \right) b_2, \quad a_3 = \left(\sum_{j=1}^3 c_j b_j \right) b_3$$

Basis: $\{e_1, e_2, e_3\} = \{e_i\}$

Note: - all terms must have same free indices

- there can be more than one free index

- same symbol cannot be used for both free and dummy index

Q: Why are these expressions meaningless?

1) $a_i = b_j$

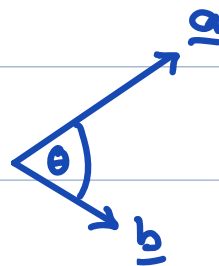
2) $a_i b_j = c_i d_j d_j$

3) $a_i b_j = c_i c_k d_k d_j + d_p c_l c_l d_q$

4) $a_i = b_k c_k d_k e_i$

Scalar product $\underline{a}, \underline{b} \in \mathcal{V}$

$$\underline{a} \cdot \underline{b} = |\underline{a}| |\underline{b}| \cos \theta \quad \theta \in [0, \pi]$$



$$\underline{a} \cdot \underline{b} = 0 \quad \underline{a} \perp \underline{b}$$

$$\underline{a} \cdot \underline{a} = |\underline{a}|^2$$

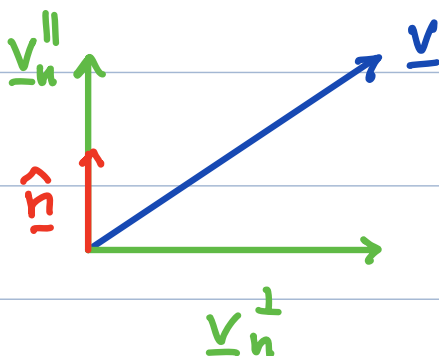
$$\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a} \quad \text{commutative}$$

Projection: $\hat{n}$ = unit vector

$$\underline{v} = \underline{v}_n^{\parallel} + \underline{v}_n^{\perp}$$

$$\underline{v}_n^{\parallel} = (\underline{v} \cdot \hat{n}) \hat{n}$$

$$\underline{v}_n^{\perp} = \underline{v} - \underline{v}_n^{\parallel}$$



$\Rightarrow$ components in a basis $\{\underline{e}_i\}$

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$v_1 = \underline{v} \cdot \underline{e}_1$$

$$v_2 = \underline{v} \cdot \underline{e}_2$$

$$v_3 = \underline{v} \cdot \underline{e}_3$$

Kronecker Delta

To any frame $\{\underline{e}_i\}$ we associate

$$\delta_{ij} = \underline{e}_i \cdot \underline{e}_j = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

$\Rightarrow$ orthonormal basis

δ_{ij} expresses scalar product in index notation

Properties: $\delta_{ij} = \delta_{ji}$ symmetry
 $\underline{e}_i = \delta_{ij} \underline{e}_j$ transfer property

Example: Projection on basis

$$\underline{u} = u_i \underline{e}_i$$

$$\underline{u} \cdot \underline{e}_j = (u_i \underline{e}_i) \cdot \underline{e}_j = u_i \underline{e}_i \cdot \underline{e}_j = u_i \delta_{ij} = u_j$$

Scalar product: $\underline{a} = a_i \underline{e}_i$ $\underline{b} = b_j \underline{e}_j$

$$\begin{aligned} \underline{a} \cdot \underline{b} &= (a_i \underline{e}_i) \cdot (b_j \underline{e}_j) = a_i b_j \underline{e}_i \cdot \underline{e}_j \\ &= a_i b_j \delta_{ij} \\ &= a_i b_i = a_j b_j \end{aligned}$$

$\Rightarrow \delta_{ij}$ disappears due to transfer property!

Vector product

$$\underline{a}, \underline{b} \in \mathcal{V}$$

$$\underline{a} \times \underline{b} = |\underline{a}| |\underline{b}| \sin \theta \hat{n}$$

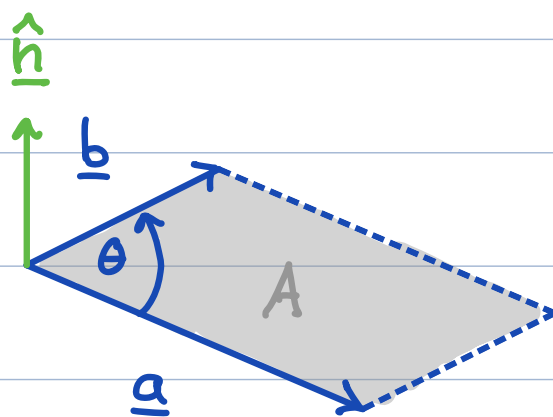
$$\theta \in [0, \pi]$$

$\hat{n}$ unit vector $\perp$ to $\underline{a}$ & $\underline{b}$

(right hand rule)

$|\underline{a} \times \underline{b}| = \text{area of parallelogram}$

Q: Significance of $\underline{a} \times \underline{b} = \underline{0}$?



Note: $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a} \Rightarrow$ not commutative

Permutation symbol (Levi-Civita)

vector product in index notation

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \in \{123, 231, 312\} \text{ even perm.} \\ -1 & \text{if } ijk \in \{321, 213, 132\} \text{ odd perm.} \\ 0 & \text{repeated index} \end{cases}$$

Flipping any 2 indices changes sign

$$\epsilon_{ijk} = -\epsilon_{kji} = -\epsilon_{jik} = -\epsilon_{ikj}$$

Invariant under cyclic permutation

$$\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki}$$

Relation to frame $\{\underline{e}_i\}$

$$\underline{e}_i \times \underline{e}_j = \epsilon_{ijk} \underline{e}_k$$

Vector product: $\underline{a} \times \underline{b} = \underline{c}$

$$\underline{a} = a_i \underline{e}_i \quad \underline{b} = b_j \underline{e}_j \quad \underline{c} = c_k \underline{e}_k$$

$$\underline{a} \times \underline{b} = (a_i \underline{e}_i) \times (b_j \underline{e}_j) = a_i b_j (\underline{e}_i \times \underline{e}_j)$$

$$= a_i b_j \epsilon_{ijk} \underline{e}_k = \underline{c} =$$
$$= \underbrace{a_i b_j \epsilon_{ijk}}_{c_k} \underline{e}_k$$

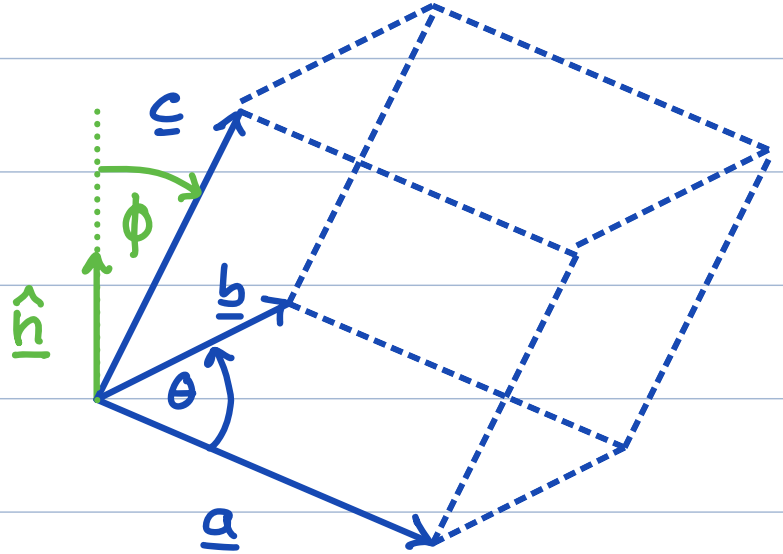
$$\Rightarrow \boxed{c_k = \epsilon_{ijk} a_i b_j}$$

Triple scalar product

$$\underline{a}, \underline{b}, \underline{c} \in \mathbb{R}^3$$

$$(\underline{a} \times \underline{b}) \cdot \underline{c} = |\underline{a}| |\underline{b}| |\underline{c}| \sin \theta \cos \phi$$

$\Rightarrow$ Volume of the
parallel piped



$$(\underline{a} \times \underline{b}) \cdot \underline{c} = 0 \Rightarrow \text{linearly dependent}$$

$$(\underline{a} \times \underline{b}) \cdot \underline{c} > 0 \Rightarrow \text{right handed}$$

$$(\underline{a} \times \underline{b}) \cdot \underline{c} < 0 \Rightarrow \text{left handed}$$

Cartesian reference frame

right handed orthonormal basis $\{\underline{e}_i\}$

$$\Rightarrow (\underline{e}_1 \times \underline{e}_2) \cdot \underline{e}_3 = 1$$

Relation to Levi-Civita

$$\epsilon_{ijk} = (\underline{e}_i \times \underline{e}_j) \cdot \underline{e}_k$$

$$\begin{aligned} \text{Proof: } \epsilon_{ijk} &= (\underline{e}_i \times \underline{e}_j) \cdot \underline{e}_k \\ &= \epsilon_{ijl} \underline{e}_l \cdot \underline{e}_k \\ &= \epsilon_{ijl} \delta_{lk} = \epsilon_{ijk} \quad \checkmark \end{aligned}$$

$$\text{Use: } \underline{a} = a_i \underline{e}_i, \quad \underline{b} = b_j \underline{e}_j, \quad \underline{c} = c_k \underline{e}_k$$

$$\begin{aligned} (\underline{a} \times \underline{b}) \cdot \underline{c} &= ((a_i \underline{e}_i) \times (b_j \underline{e}_j)) \cdot (c_k \underline{e}_k) \\ &= a_i b_j c_k \underbrace{(\underline{e}_i \times \underline{e}_j) \cdot \underline{e}_k}_{\epsilon_{ijk}} \\ &= \epsilon_{ijk} a_i b_j c_k \end{aligned}$$

$$(\underline{a} \times \underline{b}) \cdot \underline{c} = \epsilon_{ijk} a_i b_j c_k$$

$\Rightarrow$ Invariant under cyclic permutation

$$(\underline{a} \times \underline{b}) \cdot \underline{c} = (\underline{c} \times \underline{a}) \cdot \underline{b} = (\underline{b} \times \underline{c}) \cdot \underline{a}$$

Relation ship to determinant

$$\text{matrix } [\underline{a} \ \underline{b} \ \underline{c}] = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$(\underline{a} \times \underline{b}) \cdot \underline{c} = \det([\underline{a} \ \underline{b} \ \underline{c}])$$

• commutative cyclic

$$(\underline{a} \times \underline{b}) \cdot \underline{c} \stackrel{\downarrow}{=} \underline{c} \cdot (\underline{a} \times \underline{b}) \stackrel{\downarrow}{=} \underline{a} \cdot (\underline{b} \times \underline{c})$$

$$\underline{b} \times \underline{c} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \underline{e}_1(b_2 c_3 - b_3 c_2) - \underline{e}_2(b_1 c_3 - b_3 c_1) + \underline{e}_3(b_1 c_2 - b_2 c_1)$$

taking dot product with $\underline{a}$ replaces first row

$$\Rightarrow \underline{a} \cdot (\underline{b} \times \underline{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \det([\underline{a} \ \underline{b} \ \underline{c}])$$

determinants $\Rightarrow$ volumes

Frame Identities

frame $\{\underline{e}_i\}$ = orthonormal basis

$$\boxed{\underline{e}_i = \delta_{ij} \underline{e}_j} \quad \text{and} \quad \boxed{\underline{e}_i \times \underline{e}_j = \epsilon_{ijk} \underline{e}_k}$$

These are the two basic identities relating the basis vectors of any cartesian reference frame.

$\Rightarrow$ used continuously in index manipulation.

Example of index manipulation:

$$(\underline{a} \cdot \underline{b}) \underline{c} + \underline{b} = ?$$

$$\underline{a} = a_i \underline{e}_i, \quad \underline{b} = b_j \underline{e}_j, \quad \underline{c} = c_k \underline{e}_k$$

1) substitute.

$$(a_i \underline{e}_i \cdot b_j \underline{e}_j) c_k \underline{e}_k + b_j \underline{e}_j = ?$$

2) pull coefficients out to front

$$= a_i b_j c_k (\underline{e}_i \cdot \underline{e}_j) \underline{e}_k + b_j \underline{e}_j$$

3) simplify using definition of dot product

$$= a_i b_j c_k \delta_{ij} \underline{e}_k + b_j \underline{e}_j$$

4) use transfer property

$$= a_i b_i c_k \underline{e}_k + b_j \underline{e}_j$$

5) Rename indices to collect terms

$$= a_i b_i c_k \underline{e}_k + b_k \underline{e}_k$$

$$= \underline{(a_i b_i c_k + b_k) \underline{e}_k}$$