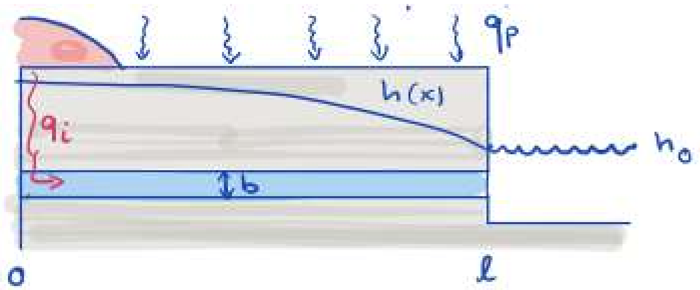
Highlands Aquifer with precipitation and polar recharge

The high heat flow on early Mars may have lead to basal melting of the ice caps. This has been estimated to introduce > 10 km^3/yr into the Martian Highlands aquifer. Clifford and Parker (2001) modeled this as an inflow through the boundary condtion.



Assuming (for now) a steady linear confined aquifer we have the following model equations:

PDE: $-\frac{\mathrm{d}}{\mathrm{d}x}\left[bK\frac{\mathrm{d}h}{\mathrm{d}x}\right] = q_p$ on $x \in [0,l]$

BC's: $q_i = -K\frac{\mathrm{d}h}{\mathrm{d}x}\Big|_0 \Longrightarrow \frac{\mathrm{d}h}{\mathrm{d}x}\Big|_0 = -\frac{q_i}{K}$ and $h(l) = h_o$

Dimensionless equations

Choosing the same dimesionless paramters as in the case without polar recharge

$x' = \frac{x}{l}$ and $h' = \frac{h-h_o}{q_p l^2/(bK)}$

we have the dimensionless problem

PDE: $-\frac{\mathrm{d}^2h'}{\mathrm{d}x'^2} = 1$ on $x' \in [0,1]$

BC's: $\frac{\mathrm{d}h'}{\mathrm{d}x'}\Big|_0 = \Pi$ and $h'(1) = 0$.

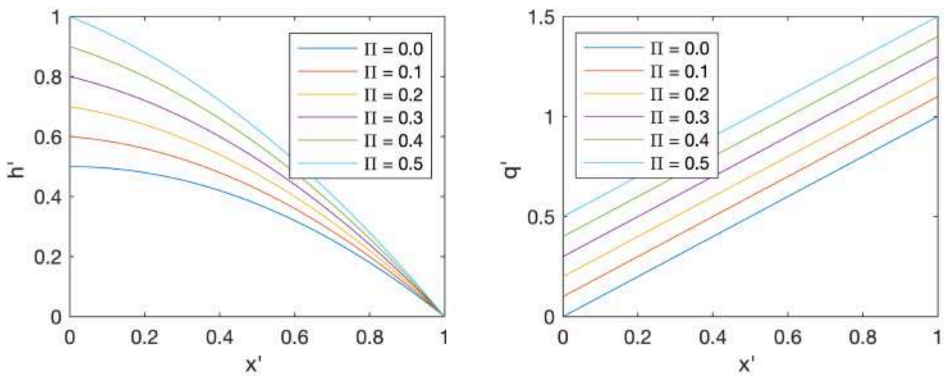
which has one dimensionless parameter $\Pi = q_i b/(q_p l)$. The analytic solution is obtained by integrating twice and given by

$h' = \frac{1}{2}(1-x'^2) + \Pi(1-x')$ and $q' = x' + \Pi$.

This family of solutions for differen values of Π is shown in the figure.

```
clear
q_ana = @(x,Pi) x + Pi;
h_ana = @(x,Pi) .5*(1-x.^2)+Pi*(1-x);

x_ana = linspace(0,1,1e2);
```

```
subplot 121
plot(x_ana,h_ana(x_ana,0)), hold on
plot(x_ana,h_ana(x_ana,0.1))
plot(x_ana,h_ana(x_ana,0.2))
plot(x_ana,h_ana(x_ana,0.3))
plot(x_ana,h_ana(x_ana,0.4))
plot(x_ana,h_ana(x_ana,0.5))
xlabel 'x'' ', ylabel 'h'' '
legend('\Pi = 0.0','\Pi = 0.1','\Pi = 0.2','\Pi = 0.3','\Pi = 0.4','\Pi = 0.5')
pbaspect([1 .8 1])

subplot 122
plot(x_ana,q_ana(x_ana,0)), hold on
plot(x_ana,q_ana(x_ana,0.1))
plot(x_ana,q_ana(x_ana,0.2))
plot(x_ana,q_ana(x_ana,0.3))
plot(x_ana,q_ana(x_ana,0.4))
plot(x_ana,q_ana(x_ana,0.5))
xlabel 'x'' ', ylabel 'q'' '
legend('\Pi = 0.0','\Pi = 0.1','\Pi = 0.2','\Pi = 0.3','\Pi = 0.4','\Pi = 0.5','locatio
pbaspect([1 .8 1])
```

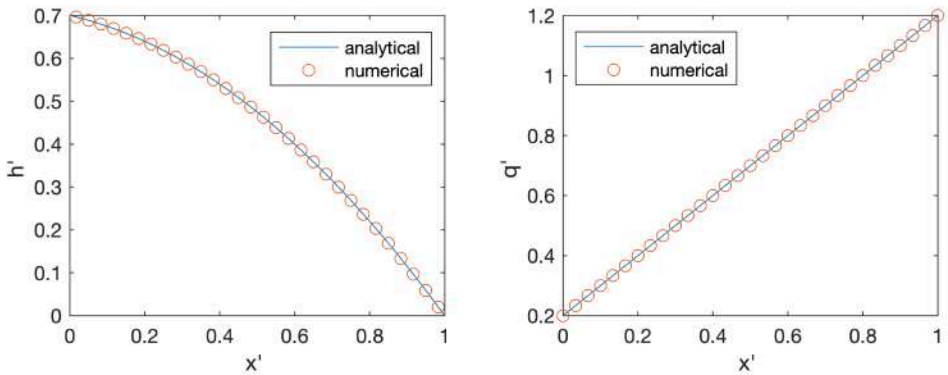


Numerical solution

Here we apply the flux Pi on the left hand boundary of the domain. Similar to the Dirichlet BC we specify the location and flux of the Neumann BC with the vectors `Grid.dof_neu` and `Grid.qb`, respectively.

Note, that `fn`, the r.h.s. vector arising from the Neumann BC, must be added to `fs` in the input to `solve_lbvp.m`.

```
Pi = 0.2;
Grid.xmin = 0; Grid.xmax = 1; Grid.Nx = 30;
Grid = build_grid(Grid);
[D,G,I] = build_ops(Grid);
L = -D*G; fs = ones(Grid.Nx,1);

BC.dof_dir = Grid.dof_xmax;
BC.dof_f_dir = Grid.dof_f_xmax;
BC.g = h_ana(Grid.xc(Grid.dof_xmax),Pi);

BC.dof_neu = Grid.dof_xmin;
BC.dof_f_neu = Grid.dof_f_xmin;
BC.qb = Pi;
[B,N,fn] = build_bnd(BC,Grid,I);

h = solve_lbvp(L,fs+fn,B,BC.g,N);
q = comp_flux(D,1,G,h,fs,Grid,BC);

figure
subplot 121
plot(x_ana,h_ana(x_ana,Pi)), hold on
plot(Grid.xc,h,'o')
xlabel 'x'' ', ylabel 'h'' '
legend('analytical','numerical')
pbaspect([1 .8 1])

subplot 122
plot(x_ana,q_ana(x_ana,Pi)), hold on
plot(Grid.xf,q,'o')
xlabel 'x'' ', ylabel 'q'' '
legend('analytical','numerical','location','northwest')
pbaspect([1 .8 1])
```



Discrete conservation

Our equations are based on the balance of fluid mass and hence any convergent (=functioning) numerical solution should satisfy water balance in the limit of very fine resolution. By discrete conservation we mean the property of a numerical scheme to **satisfy the conservation exactly even on an arbitrarily coarse grid**! We can demonstrate this with the problem at hand. The outflow of groundwater into the northern ocean must exactly balance the input due to rain and polar recharge.

flux into ocean = flux due to polar recharge + precipitation added

In the analytic solution:

Flux due to polar recharge: $q_r' = \Pi$

Water added by precipitation: $q_p' = 1$

Flow rate of water into the ocean in: $q_o' = q'(1) = 1 + \Pi$

```
Pi = 0.2;
Grid.xmin = 0; Grid.xmax = 1; Grid.Nx = 5;
Grid = build_grid(Grid);
[D,G,I] = build_ops(Grid);
L = -D*G; fs = ones(Grid.Nx,1);
```

```
BC.dof_dir = Grid.dof_xmax;
BC.dof_f_dir = Grid.dof_f_xmax;
BC.g = h_ana(Grid.xc(Grid.dof_xmax),Pi);

BC.dof_neu = Grid.dof_xmin;
BC.dof_f_neu = Grid.dof_f_xmin;
BC.qb = Pi;
[B,N,fn] = build_bnd(BC,Grid,I);

h = solve_lbvp(L,fs+fn,B,BC.g,N);
q = comp_flux(D,1,G,h,fs,Grid,BC);

q_o = q(Grid.dof_f_xmax)
```

```
q_o = 1.2000
```

```
1+Pi
```

```
ans = 1.2000
```

Even on a Grid with only 5 grid points the flux into the ocean is exactly $1 + \Pi$!