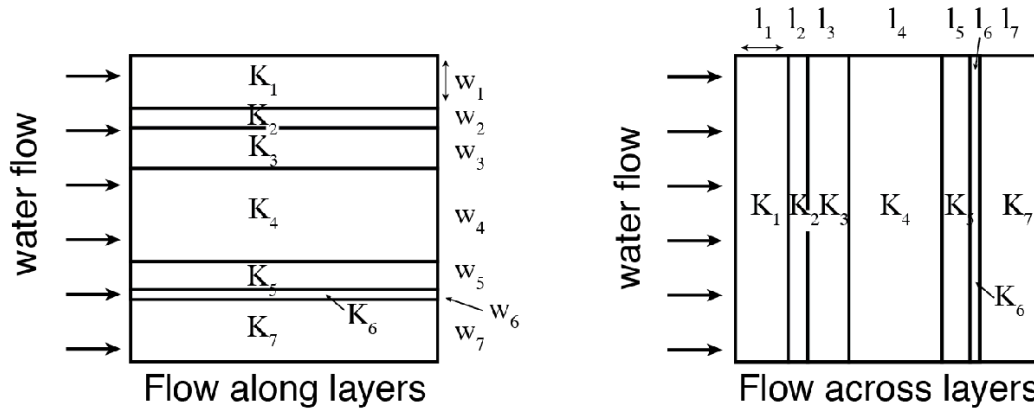


# Steady water flow across layered media

```
clear
set_demo_defaults()
```

It is instructive to consider steady groundwater flow across layered materials, because geological materials are layered at many scales. In the two limiting cases the flow is either **parallel** to the layering or **perpendicular** to the layering.



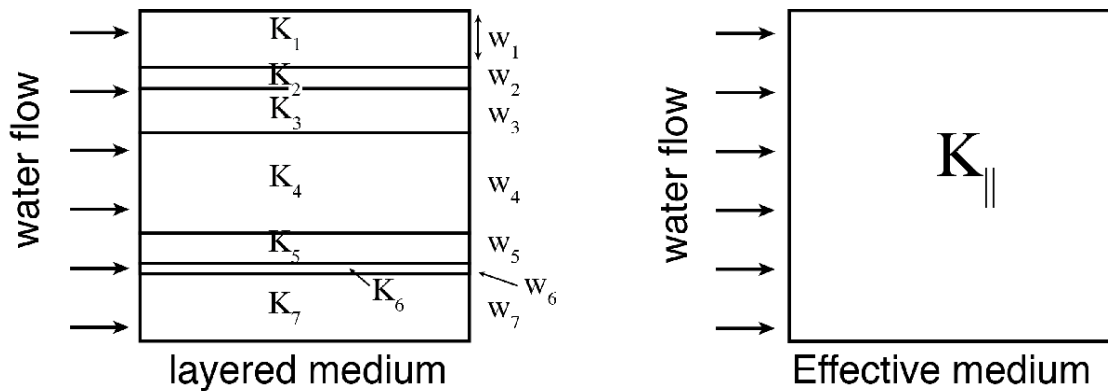
Consider a material with  $M$  layers, where the  $n$ -th layer has hydraulic conductivity,  $K_n$ , and width or length,  $w_n$  or  $l_n$ , depending on flow direction. The total width or length of the material are therefore

$$W = \sum_{n=1}^N w_n \text{ and } L = \sum_{n=1}^N l_n.$$

## Effective conductivities

In many materials the layering is too small scale, relative to the domain of interest, to represent it directly in the numerical discretization. Hence it is important to be able to represent the layering with effective upscaled hydraulic conductivities. As we will see below, these effective conductivities will depend on direction, relative to the layering.

### Flow along the layers



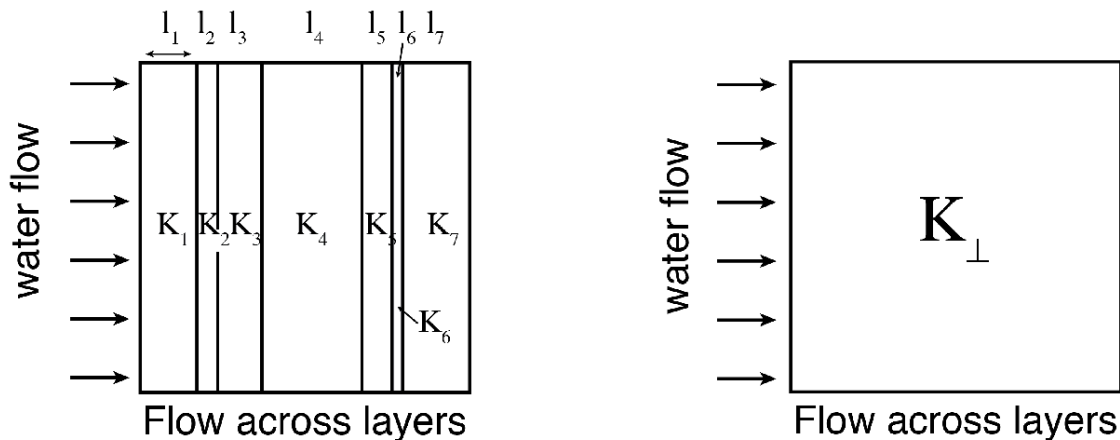
We are looking for an

effective thermal conductivity,  $\kappa_{\parallel}$ , so that the total flow through the medium,  $Q = Wq = -W \kappa_{\parallel} \frac{\Delta h}{L}$ , is identical

to the total flow obtained for the layered medium with the same head gradient,  $\Delta h/\Delta L$ . In the layered medium the total flow rate is the sum of the flow rates in each layer, because the the flow is parallel to the layers:  $Q = \sum_{n=1}^N w_n q_n$ , where  $q_n$  is the flux in the n-th layer. Equating these two expressions we obtain

$W K_{\parallel} = \sum_{n=1}^N w_n K_n$ , so that the effective flux is given by  $K_{\parallel} = \sum_{n=1}^N \frac{w_n}{W} K_n$ . Hence, the effective hydraulic conductivity for flow along layers is the fractional width weighted **arithmetic mean** of the conductivities of the individual layers.

### Flow across the layers



Now let's develop an analogous expression the effective hydraulic conductivity,  $K_{\perp}$ , for flow across the layers. In this case, the flux across each layer must be identical,  $q = q_n = -K_n \Delta h_n / l_n$ , but the head gradient across each layer  $\Delta h_n / l_n$ , will be different. The total head drop and the total medium length are then given by

$$\Delta h = \sum_{n=1}^N \Delta h_n, \text{ and } L = \sum_{n=1}^N l_n.$$

Substituting the first into the expression for the flux in terms of the effective conductivity we get

$$q = -K_{\perp} \frac{\sum_{n=1}^N \Delta T_n}{L}$$

now we use the expression for the flux across each layer to eliminate the head drops and obtain

$$qL = -K_{\perp} \sum_{n=1}^N \left( -\frac{q l_n}{K_n} \right),$$

where  $q$  cancels and we re-arrange to obtain the following expression the effective hydraulic conductivity across layers:

$$K_{\perp} = \frac{1}{\sum_{n=1}^N \frac{l_n/L}{K_n}}$$

This is called the harmonic average, and again it is weighted by the length fraction,  $l_n/L$ , of each layer.

## General Power-law average

These two averages are special cases of the general power-law average, given by

$$\langle K \rangle_p = \left( \frac{1}{N} \sum_{n=1}^N (K_n)^p \right)^{1/p}$$

where the arithmetic average is recovered for  $p=1$  and the harmonic average for  $p=-1$ .

(Note to self - still need to work out the weighting!)

## Numerical examples

Note: This requires 2D operators, which you don't have yet.

```
kappa_vec = [30;100;30;75;350; 5;120];
dist_vec = [10; 35; 5;15; 5;20; 10];
Grid.xmin = 0; Grid.xmax = 1; Grid.Nx = 100;
Grid.ymin = 0; Grid.ymax = 1; Grid.Ny = 100;
Grid = build_grid(Grid);
[D,G,I] = build_ops(Grid);
L = @(Kd) -D*Kd*G; fs = zeros(Grid.N,1);
BC.dof_dir = [Grid.dof_xmin;Grid.dof_xmax];
BC.dof_f_dir = [Grid.dof_f_xmin;Grid.dof_f_xmax];
BC.g = [ones(Grid.Ny,1);zeros(Grid.Ny,1)];
BC.dof_neu = []; BC.dof_f_neu = []; BC.qb = [];
[B,N,fn] = build_bnd(BC,Grid,I);
```

First let's look at flow along the layers

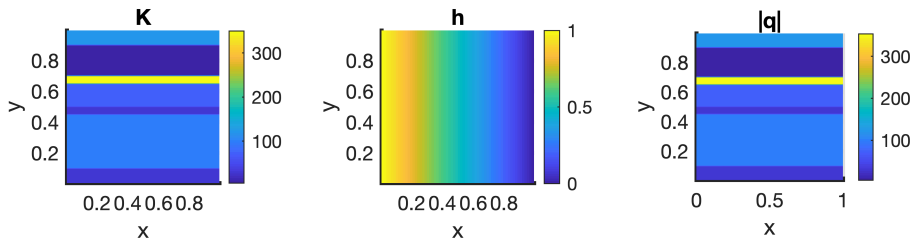
```
[Kd_x,K_x] = make_layered_medium(kappa_vec,dist_vec,Grid,'x-dir');
u = solve_lbvp(L(Kd_x),fs+fn,B,BC.g,N);
q = comp_flux(D,Kd_x,G,u,fs,Grid,BC);

subplot 131
[Xc,Yc] = meshgrid(Grid.xc,Grid.yc);
surf(Xc,Yc,K_x), shading interp, view(2), colorbar
xlabel 'x', ylabel 'y', title 'K'
axis square tight

subplot 132
[Xc,Yc] = meshgrid(Grid.xc,Grid.yc);
surf(Xc,Yc,reshape(u,Grid.Ny,Grid.Nx)), shading interp, view(2), colorbar
xlabel 'x', ylabel 'y', title 'h'
axis square tight

subplot 133
[Xx,Yx] = meshgrid(Grid.xf,Grid.yc);
```

```
surf(Xx,Yx,reshape(q(1:Grid.Nfx),Grid.Ny,Grid.Nx+1)), shading interp, view(2), colorbar
xlabel 'x', ylabel 'y', title '|q|'
axis square tight
```



Now compare the flow computed numerically on the layered medium with the effective expression

```
Qnum = q(Grid.dof_f_xmin)'*Grid.A(Grid.dof_f_xmin)
```

```
Qnum = 82.0707
```

```
kappa_eff = (dist_vec/100)'*kappa_vec;
Qeff = kappa_eff*1/(1-Grid.dx)
```

```
Qeff = 82.0707
```

Then look at flow across the layers

```
[Kd_y,K_y] = make_layered_medium(kappa_vec,dist_vec,Grid,'y-dir');
u = solve_lbvp(L(Kd_y),fs+fn,B,BC.g,N);
q = comp_flux(D,Kd_y,G,u,fs,Grid,BC);
```

```
subplot 131
[Xc,Yc] = meshgrid(Grid.xc,Grid.yc);
surf(Xc,Yc,K_y), shading interp, view(2), colorbar
xlabel 'x', ylabel 'y', title 'K'
```

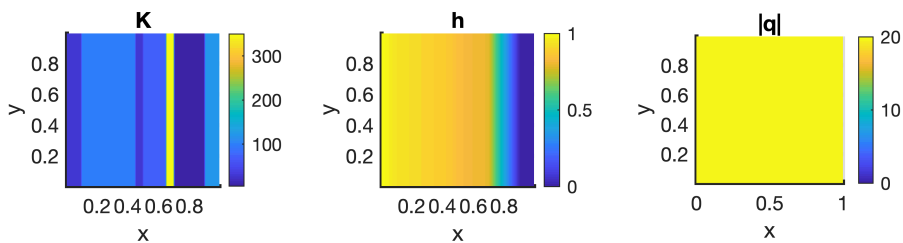
```
axis square tight
```

```
subplot 132
```

```
[Xc,Yc] = meshgrid(Grid.xc,Grid.yc);  
surf(Xc,Yc,reshape(u,Grid.Ny,Grid.Nx)), shading interp, view(2), colorbar  
xlabel 'x', ylabel 'y', title 'h'  
axis square tight
```

```
subplot 133
```

```
[Xx,Yx] = meshgrid(Grid.xf,Grid.yc);  
surf(Xx,Yx,reshape(q(1:Grid.Nfx),Grid.Ny,Grid.Nx+1)), shading interp, view(2), colorbar  
caxis([0 20])  
xlabel 'x', ylabel 'y', title '|q|'  
axis square tight
```



Now compare the flux computed directly on the layers medium with the effective expression

```
qnum = q(1)
```

```
qnum = 19.5054
```

```
kappa_eff = 100./sum((dist_vec./kappa_vec));  
qeff = kappa_eff*1/(1-Grid.dx)
```

```
qeff = 19.6227
```

## Auxillary functions

```
function [Kd,K] = make_layered_medium(kappa,dist,Grid,orientation)
if length(kappa) ~=length(dist); error('kappa and dist not same length\n'); end
K = ones(Grid.Ny,Grid.Nx);
if strcmp(orientation,'x-dir')
    K([1:dist(1)],:) = kappa(1);
    for i=2:length(kappa)
        K(sum(dist(1:i-1))+1:sum(dist(1:i)),:) = kappa(i);
    end
else
    K(:, [1:dist(1)]) = kappa(1);
    for i=2:length(kappa)
        K(:,sum(dist(1:i-1))+1:sum(dist(1:i))) = kappa(i);
    end
end

Kd = comp_mean(K,-1,1,Grid);
end
```