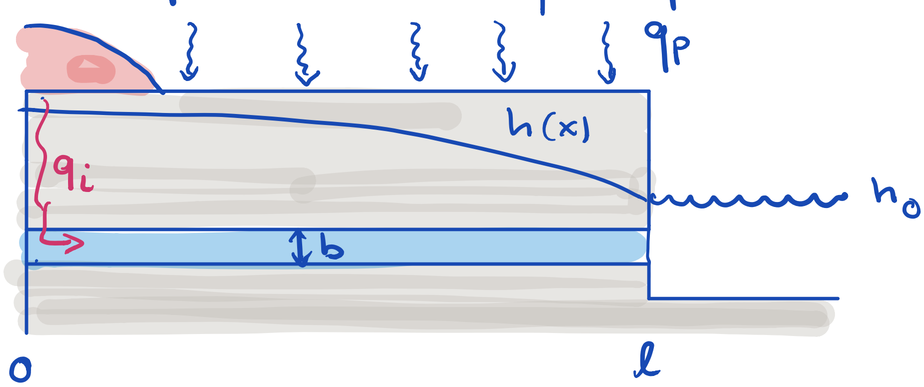
Example 2: Aquifer with polar basal melting

The high heat flow on early mars may have lead to basal melting of the ice caps. This has been estimated to introduce $> 10\ km^3/yr$ into the Martian Highlands aquifer.

Clifford & Parker (2001) modeled this as a flux through the boundary.

$\Rightarrow$ two fluid sources: precipitation & polar melting



PDE: $$-\frac{d}{dx}\left[b K \frac{dh}{dx}\right] = q_p \qquad x \in [0,\ell]$$

BC: $$q_i = -K \left.\frac{dh}{dx}\right|_0 \quad \Rightarrow \quad \left.\frac{dh}{dx}\right|_0 = -\frac{q_i}{K}$$

$$h(\ell) = h_b$$

Non-dimensionalization

Char. scales: $x' = \frac{x}{\ell}$ $\quad h' = \frac{h - h_0}{h_c}$

PDE: $-\frac{d^2 h'}{dx'^2} = \frac{q_p \ell^2}{b K h_c}$ $\quad x' \in [0, 1]$

BC: $\frac{dh'}{dx'} = -\frac{q_i \ell}{K h_c}$ $\quad h'(1) = 0$

$\Rightarrow$ Two dimensionless groups:

I) $\frac{q_p \ell^2}{b K h_c} = 1 \quad \Rightarrow \quad h_c^{I} = \frac{q_p \ell^2}{b K}$

II) $\frac{q_i \ell}{K h_c} = 1 \quad \Rightarrow \quad h_c^{II} = \frac{q_i \ell}{K}$

choose the scale associated with the dominant process, here precipitation

$h_c = h_c^{I} = \frac{q_p \ell^2}{b K}$ (as before)

$\Rightarrow$ dimensionless governing parameter

$$\Pi = \frac{q_i \ell}{K h_c} = \frac{q_i b}{q_p \ell}$$

Dimensionless equations

$$\text{PDE:}\quad -\frac{d^2h'}{dx'^2} = 1, \qquad x' \in [0,1]$$

$$\text{BC:}\quad \left.\frac{dh'}{dx'}\right|_0 = -\Pi, \qquad h'(1) = 0$$

Interpretation of Π:

surface area: $A_s = lw$

x-sectional : $A_c = bw$

$$\Pi = \frac{q_i b}{q_p l} = \frac{q_i b w}{q_p l w} = \frac{q_i A_c}{q_p A_s} = \frac{Q_i}{Q_p}$$

$\Rightarrow$ Π is the ratio of the flow rate due to polar basal melting, Q_i, to the flow rate due to precipitation, Q_p.

Analytic solution:

Integrate once: $-\frac{dh'}{dx'} = x' + c_1$

use 1st BC: $-\left.\frac{dh'}{dx'}\right|_0 = 0 + c_1 = \Pi \quad \Rightarrow \quad c_1 = \Pi$

Integrate again: $-h' = \frac{x'^2}{2} + \Pi x' + c_2$

use 2nd BC: $0 = \frac{1}{2} + \Pi + c_2 = 0 \Rightarrow c_2 = -\frac{1}{2} - \Pi$

$$\boxed{\begin{aligned} h' &= \tfrac{1}{2}(1 - x'^2) + \Pi(1 - x) \\ q' &= x' + \Pi \end{aligned}}$$

What is scale for q?

It is induced by other scales: $q' = \frac{q}{q_c}$

Darcy: $q_c q' = -K \frac{h_c}{\ell} \frac{dh'}{d\ell'} \Rightarrow q_c = K \frac{h_c}{\ell}$

$$q' = -\frac{dh'}{d\ell'}$$