Heterogeneous Dirichlet BC's & constraints

PDE, $-\frac{d}{dx}\left(K\frac{dh}{dx}\right)=0 \qquad x\in[0,L]$

BC $\qquad h(0)=h_0 \quad h(L)=h_L \qquad h_0, h_L \in \mathbb{R}$

$\Rightarrow$ Linear system for BC's : $\underset{\uparrow}{\underline{\underline{B}}}\,\underline{h}=g$ where $g=\begin{bmatrix}h_0\\h_L\end{bmatrix}$

same as before

because $\underline{\underline{B}}\underline{h}=g$ s linear we can decompose $\underline{h}=\underline{h}_o+\underline{h}_p$,
into a homogeneous solution, $\underline{h}_o$, and a particular solution, $\underline{h}_p$.

$$\left.\begin{array}{ll}\text{homogeneous.} & \underline{\underline{B}}\,\underline{h}_o=\underline{0}\\ \text{particular:} & \underline{\underline{B}}\,\underline{h}_p=g\end{array}\right\}\ \underline{\underline{B}}(\underbrace{\underline{h}_o+\underline{h}_p}_{\underline{h}})=g$$

Note. $\underline{h}$ is unique (assuming suitable BC's)
The split of $\underline{h}$ into $\underline{h}_o$ and $\underline{h}_p$ is not unique,
but there is an obvious simplest choice!

Two questions. 1) How do we determine suitable $\underline{h}_p$?
$\rightarrow$ here we have a choice

2) Given $\underline{h}_p$ what is the associated $\underline{h}_o$?

Start with 2. Suppose we know $\underline{h}_p$

$\underline{\underline{L}}(\underline{h}_o+\underline{h}_p)=\underline{f}$ since $\underline{h}_p$ is known $\rightarrow$ rhs

$\underline{\underline{L}}\,\underline{h}_o=\underline{f}-\underline{\underline{L}}\,\underline{h}_p$ heterogeneous BC's add source term $\underline{f}_D=-\underline{\underline{L}}\,\underline{h}_p$

To solve we project into Null space of $\underline{\underline{B}}$ similar to homogeneous case.

$$\underbrace{\underline{\underline{N}}^T\underline{\underline{L}}\,\underline{\underline{N}}}_{\underline{\underline{L}}_r}\,\underbrace{\underline{\underline{N}}^T\underline{h}_o}_{\underline{h}_{or}}=\underline{\underline{N}}^T(\underline{f}_s+\underline{f}_D)=\underbrace{\underline{\underline{N}}^T\underline{f}}_{\underline{f}_r} \quad\rightarrow\quad \boxed{\underline{\underline{L}}_r\,\underline{h}_{or}=\underline{f}_r}$$

$$\underline{h}_o=\underline{\underline{N}}\,\underline{h}_{or}$$

Finding a particular solution $\underline{h}_p$:

Note that $\underline{h}_p$ does not need to satisfy $\underline{\underline{L}}\,\underline{h}_p = \underline{f}$

that is taken care of in homog solution.

Therefore $\underline{h}_p$ simply needs to solve $\underline{\underline{B}}\,\underline{h}_p = g$

Intuition: Given that $\underline{\underline{B}}$ is composed of N_c rows of $\underline{\underline{I}}$

$\underline{h}_p$ needs to have the N_c entries of g in the right places. ($\underline{h}_p = \underline{\underline{B}}^T g$ will do just this !)

To solve we need to make $\underline{\underline{B}}\,\underline{h}_p = g$ a square system

most obvious $\underbrace{\underline{\underline{B}}^T\underline{\underline{B}}}_{Nx\,Nx}\,\underline{h}_p = \underbrace{\underline{\underline{B}}^T g}_{Nx\cdot 1}$ $\underline{\underline{B}}^T\underline{\underline{B}}$ is not invertible because it is $Nx \times Nx$ but has at most N_c^2 non-zero entries

Note this transforms Nc by Nx system to Nx by Nx system

Instead of increasing system size we should reduce it to Nc by Nc

Want to solve reduced system: $\underbrace{\underline{\underline{B}}_r}_{N_c\cdot N_c}\,\underbrace{\underline{h}_{pr}}_{N_c\cdot 1} = \underbrace{g}_{N_c\cdot 1}$

Define reduced part soln:

1) $\underbrace{\underline{h}_{pr}}_{N_c\,1} = \underbrace{\underline{\underline{B}}}_{Nc\cdot Nx}\,\underbrace{\underline{h}_p}_{Nx\,1}$ $\Rightarrow$ from eqn $\underline{\underline{B}}\,\underline{h}_p = g \Rightarrow \underline{h}_{pr} = g$

to recover $\underline{h}_p$ from $\underline{h}_{pr}$ need to solve

$\underline{\underline{B}}\,\underline{h}_p = \underline{h}_{pr}$ $\quad \underline{\underline{B}}^T\underline{\underline{B}}\,\underline{h}_p = \underline{\underline{B}}^T\underline{h}_{pr} = \underline{\underline{B}}^T g$

$\Rightarrow$ same problem as above

2) $\underline{h}_p = \underline{\underline{B}}^T\underline{h}_{pr}$ no problem recovering $\underline{h}_p$ from $\underline{h}_{pr}$

substitute into eqn $\underline{\underline{B}}(\underline{\underline{B}}^T\underline{h}_{pr}) = g$

$$\boxed{\underbrace{\underline{\underline{B}}\,\underline{\underline{B}}^T}_{N_c\,N_c}\,\underbrace{\underline{h}_{pr}}_{N_c\,1} = \underbrace{g}_{N_c\,1}}$$

BB^T is invertible and often $\underline{\underline{I}}$

in which case $\underline{h}_p = \underline{\underline{B}}^T g$