Cylindrical aquifer with polar recharge

PDE: $-\frac{1}{r}\frac{d}{dr}\left[r\, b K \frac{dh}{dr}\right] = 0 \qquad r \in [r_p, \ell]$

BC: $q_i = -K \frac{dh}{dr}\Big|_{r_p} \qquad h(\ell) = h_0$

Non-dimensionalize: $r' = \frac{r}{\ell} \qquad h' = \frac{h - h_0}{h_c}$

substitute: $-\frac{1}{\ell^2}\frac{1}{r'}\frac{d}{dr'}\left[\frac{\ell}{\ell}\, bK h_c r' \frac{dh'}{dr'}\right] = 0 \qquad r' \in \left[\frac{r_p}{\ell}, 1\right]$

$\Rightarrow -\frac{d}{dr}\left[r' \frac{dh'}{dr'}\right] = 0 \qquad r \in [\rho, 1] \qquad \rho = \frac{r_p}{\ell}$

BC: $q_i = -K \frac{h_c}{\ell}\frac{dh'}{dr'}\Big|_\rho \qquad \Rightarrow -\frac{dh'}{dr'}\Big|_\rho = \frac{q_i \ell}{K h_c} \qquad \boxed{h_c = \frac{q_i \ell}{K}}$

$$\boxed{\begin{aligned} &\text{PDE:} \quad -\frac{d}{dr'}\left[r' \frac{dh'}{dr'}\right] = 0 \qquad r' \in [\rho, 1] \\ &\text{BC:} \quad q' = -\frac{dh}{dr}\Big|_\rho = 1 \qquad h'(1) = 0 \end{aligned}}$$

one dimensionless group $\rho = \frac{r_p}{\ell}$ (geometric)

Analytic solution

Integrate: $-r' \frac{dh'}{dr'} = c_1$

Neumann BC at r_p: $-\rho \left.\frac{dh}{dr'}\right|_{\rho} = c_1$ $\qquad q' = -\left.\frac{dh'}{dr'}\right|_{\rho} = \frac{c_1}{\rho} = 1$

$\boxed{c_1 = \rho}$

$\Rightarrow \quad -r' \frac{dh'}{dr'} = \rho$

Integrate by parts: $-dh' = \rho \frac{dr'}{r'}$

$-h' = \rho \log(r') + c_2$

Dirichlet BC at 1: $-0 = \rho \cancel{\log(1)}^{0} + c_2 \qquad c_2 = 0$

$$\boxed{\begin{aligned} h' &= -\rho \log(r') \\ q' &= -\frac{dh'}{dr'} = \frac{\rho}{r'} \end{aligned}}$$

$\rho = \frac{r_p}{\ell}$