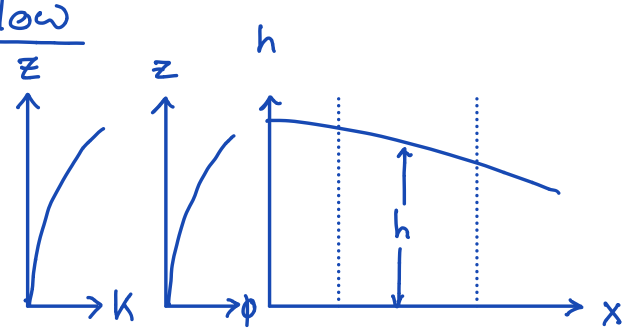
Transient unconfined flow



Let's consider an aquifer with vertical variation of both porosity $\phi(z)$ and hydraulic conductivity $K(z)$.

Any variation can be considered, but for simpicity we consider a power-law variation in both quantities:

$$\boxed{\phi = \phi_0 z^m \quad K = K_0 z^n} \qquad \text{typically} \quad \frac{n}{m} \in [2,3]$$

The balance law for fluid mass is then

$$\frac{\partial}{\partial t}\int_0^h \phi \, dz + \nabla \cdot \int_0^h \underline{q} \, dz = f_s$$

where h is the heigh of the water table

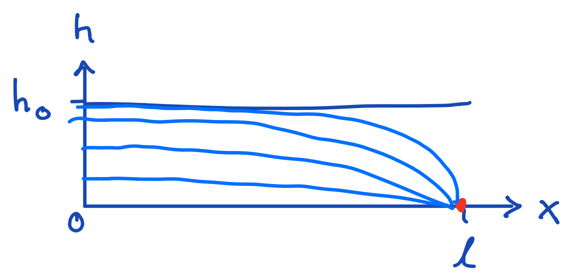
where the horizontal flux is given by the Dupuit-Forchheimer approximation: $\underline{q} = -K \nabla h$

So that we have the following governing equation for transient unconfined flow

$$\boxed{\frac{\partial h^{m+1}}{\partial t} - D_h \nabla \cdot \left[h^{n+1} \nabla h \right] = f_s}$$

This equation is non-linear both in the flux and in the accumulation term. The two exponents are typically related $n/m \in [2,3]$. This equation is variably referred to as the Boussinesq eqn or the Porous Media eqn.

Example: Drainage of an unconfined aquifer



PDE: $\frac{\partial h^{m+1}}{\partial t} + D_h \nabla \cdot [h^{n+1} \nabla h] = 0$

BC's: $\nabla h \cdot \hat{n}|_0 = 0 \qquad h(\ell) = 0$

IC: $h(x, t=0) = h_0$

This problem has both an "early" and a "late" self similar soln.

The early soln applies before the head evolution interacts with the boundary at $x=0$.

The late soln applies after the head has started interacting with the bnd at $x=0$ and $h(0,t)$ is dropping.

Here we consider the late solution.

Late self-similar solution

External length scale: $x' = \frac{x}{\ell}$

External head scale h_0 is not relevant at late time!

$\Rightarrow$ look for internal scale to help find the form of the similarity variable.

assume $h' = \frac{h}{h_c}$ $\quad t' = \frac{t}{t_c}$ substitute into PDE:

$$\frac{h_c^{m+1}}{t_c}\frac{\partial h'^{m+1}}{\partial t'} - D_h \frac{h_c^{n+2}}{\ell^2} \nabla' \cdot [h'^{n+1} \nabla h'] = 0$$

$$\frac{\partial h'^{m+1}}{\partial t'} - \frac{D_h t_c h_c^{n-m+1}}{\ell^2} \nabla' \cdot [h'^{n+1} \nabla h'] = 0$$

$\Rightarrow$ this suggests that $h_c^{n-m+1} = \frac{\ell^2}{D_h t_c}$

This would suffice to non-dimensionalize the equation but we look for a similarity variable. so we set

$$\boxed{h = \left(\frac{\ell^2}{D_h t}\right)^{\frac{1}{n-m+1}} f\left(\frac{x}{\ell}\right)}$$

Substituting this into the PDE results in the self-similar

ODE: $\frac{d}{dx'}\left(f^{n+1}\frac{df}{dx'}\right) + \frac{m+1}{n-m+1} f^{m+1} = 0$

BC: $\left.\frac{df}{dx'}\right|_0 = 0 \qquad f(1) = 0$

This non-linear ODE has no known analytic solution and must be solved numerically using the Newton-Raphson Method !

However the most important conclusions are gained without even solving the ODE. Consider the mass of GW in the aquifer, given by

$$M(t) = \rho w \int_0^{\ell} \int_0^{h} \phi \, dz \, dx = \rho w \int_0^{\ell} \frac{\phi_0}{m+1} h^{m+1} dx = \frac{\rho w \phi_0}{m+1} \int_0^{\ell} h^{m+1} dz$$

substituting the definition of the similarity variable

$$h = \left(\frac{\ell^2}{D_h t}\right)^{\frac{1}{n-m+1}} f\left(\frac{x}{\ell}\right)$$

we have

$$s = \frac{x}{\ell}$$

$$M(t) = \frac{\rho w \phi_0}{m+1} \int_0^1 \left[\left(\frac{\ell^2}{D_h t}\right)^{\frac{1}{n-m+1}} f(s)\right]^{m+1} \ell \, ds \qquad dx = \ell \, ds$$

so that

$$\boxed{M(t) = \frac{\rho w \phi_0 \ell}{m+1} \left(\frac{\ell^2}{D_h t}\right)^{\frac{m+1}{n-m+1}} \int_0^1 f^{m+1}(s) \, ds}$$

Note: $\int_0^1 f^{m+1}(s) ds$ is just a number !

$$\boxed{M(t) \sim t^{\frac{m+1}{m-n-1}}}$$

Example: $m = n = 0$ $\quad (K = \text{const.}, \ \phi = \text{const.}) \Rightarrow M \sim t^{-1} = \frac{1}{t}$

$$m = 1, \ n = 4 \quad \Rightarrow \quad M \sim t^{\frac{2}{2-4-1}} = t^{-\frac{2}{3}} = \frac{1}{t^{2/3}}$$