Flow in a ductile/viscous rock

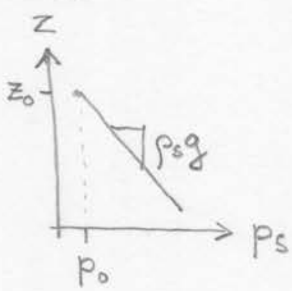
On geologic timescales ice deforms like a viscous fluid, $\eta \sim 10^{13}$ Pa s! Standard model does not work.

In particular partially molten ice can dilate if brine is overpressured and compact if it is underpressured. This basic observation is captured by the compaction relation:

$$\boxed{p_f - p_s = \xi \, \nabla \cdot \underline{v}_s} \qquad \text{constitutive law}$$

ξ = effective bulk viscosity of partially molten ice

$p_s = p_0 + \rho_s g (z_0 - z)$ solid pressure (mean stress)



Overpressure: $p = p_f - p_s$

$\Rightarrow$ determines deformation of ice

Reformulate Darcy's law in terms of overpressure

$$q_r = -\frac{k}{\mu}\left(\nabla p_f + \rho_f g \hat{z}\right) = -\frac{k}{\mu_f}\Big(\underbrace{\nabla p_f - \nabla p_s}_{\nabla p} + \underbrace{\nabla p_s}_{-\rho_s g \hat{z}} + \rho_f g \hat{z}\Big)$$

$$\boxed{q_r = -\frac{k}{\mu_f}\left(\nabla p + \Delta\rho g \hat{z}\right)} \qquad \Delta\rho = \rho_f - \rho_s > 0$$

Substitute compaction relation and Darcy's law into two phase continuity equation

$$\nabla \cdot (q_r + \underline{v}_s) = -\frac{\Delta\rho}{\rho_f \rho_s} \Gamma$$

$$\nabla \cdot q_r + \nabla \cdot \underline{v}_s = -\frac{\Delta\rho}{\rho_f \rho_s} \Gamma$$

$$\boxed{-\nabla \cdot \left(\frac{k}{\mu_f} (\nabla p + \Delta\rho g \hat{z}) \right) + \frac{p}{\xi} = -\frac{\Delta\rho}{\rho_f \rho_s} \Gamma}$$

Notes: - modified Helmholtz equation
- instantaneous, ie. not timedependent
- over pressure, p, is the unknown.

Porosity evolution

Porosity is not a simple function of p
⇒ need to solve porosity evolution equation
use solid mass balance and substitute compaction relation

$$\boxed{\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \underline{v}_s) = \frac{\Gamma}{\rho_s} + \frac{p}{\xi}}$$

- porosity moves with the solid velocity
- created by melting
- created by overpressure

⇒ need to determine solid velocity

Solid velocity field

Strictly we have to solve compressible Stokes eqn for the solid velocity field. Here we will use an approximation that is valid in $\phi \ll 1$ limit.

Helmholtz decomposition of solid velocity:

$$\underline{v}_s = -\nabla \mathcal{U} + \underbrace{\nabla \times \underline{a}}_{\text{shear}}$$

$\mathcal{U}$ = salar potential

$\underline{a}$ = vector potential

Neglect the shear component $\Rightarrow$ $\underline{v}_s = -\nabla \mathcal{U}$

Substitute into compaction equation:

$$p = \xi \, \nabla \cdot \underline{v}_s = -\xi \, \nabla \cdot \nabla \mathcal{U}$$

$$\boxed{-\nabla^2 \mathcal{U} = \frac{p}{\xi}}$$ Poisson equation

Hence to solve for flow in ductile rocks we need to solve the following set of PDE's

$$\boxed{\begin{aligned}
&1) \quad \frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \underline{v}_s) = \frac{\Gamma}{\rho_s} + \frac{p}{\xi} \\
&2) \quad -\nabla \cdot \left(\frac{k}{\mu} (\nabla p + \Delta \rho g \hat{z}) \right) + \frac{p}{\xi} = -\frac{\Delta \rho}{\rho_f \rho_s} \Gamma \\
&3) \quad -\nabla^2 \mathcal{U} = \frac{p}{\xi}
\end{aligned}}$$

where: $\underline{v}_s = -\nabla \mathcal{U}$

$k = k_0 \phi^n \qquad n \in [2, 3]$

$\xi = \frac{\eta}{\phi^m} \qquad m \in [0, 1]$