Flow in an elastic rock

Standard case considered in civil / petroleum engineering.

Bulk rock compressibility: $c_r = -\frac{1}{V_T}\frac{dV_T}{d\sigma'}\Big|_T \sim 10^{-8}\,\frac{1}{\mathrm{Pa}}$

V_T = total volume $= V_f + V_s$

σ' = effective stress

Terzaghi's principle: $\underset{\substack{\uparrow \\ \text{solid}}}{\sigma_T = \sigma'} + \underset{\substack{\uparrow \\ \text{fluid}}}{p_f}$

Volumetric strain rate: $\dot{\varepsilon}_{vol} = \frac{1}{V_T}\frac{dV_T}{dt} = \nabla \cdot \underline{v}_s$

from compressibility: $\frac{1}{V_T}\frac{dV_T}{dt} = -c_r\frac{d\sigma'}{dt} = c_r\left(\frac{\partial p_f}{\partial t} - \frac{\partial \sigma_T}{\partial t}\right)$

$$\Rightarrow \quad \boxed{\nabla \cdot \underline{v}_s = c_r\left(\frac{\partial p_f}{\partial t} - \frac{\partial \sigma_T}{\partial t}\right)}$$

substitute into continuity equation together with Darcy's law

$$\nabla \cdot (\underline{q}_f + \underline{v}_s) = -\frac{\Delta\rho}{\rho_f \rho_s}\Gamma$$

$$-\nabla \cdot \left(\frac{k}{\mu}(\nabla p_f + \rho_f g \hat{z})\right) + c_r\frac{\partial p_f}{\partial t} - c_r\frac{\partial \sigma_T}{\partial t} = -\frac{\Delta\rho}{\rho_f \rho_s}\Gamma$$

Groundwater flow equation:

$$\boxed{c_r\frac{\partial p_f}{\partial t} - \nabla \cdot \left(\frac{k}{\mu}(\nabla p_f + \rho_f g \hat{z})\right) = -\frac{\Delta\rho}{\rho_f \rho_s} + c_r\frac{\partial \sigma_T}{\partial t}}$$

Notes:
- parabolic equation (diffusion type)
- transient, i.e. time dependent
- p_f is the unknown to be solved for
- σ_T is a known loading rate

porosity change in an elastic rock is only a function of pressure. $\rightarrow$ don't need an evolution equation

$$c_r = -\frac{1}{V_T}\frac{dV_T}{d\sigma'}\bigg|_T \qquad \text{where } V_T = V_f + V_s$$

assume solid grains are incompressible $V_s = \text{const}$

$V_s = (1-\phi)V_T$ so that $dV_s = (1-\phi)dV_T - V_T d\phi = 0$

$$\Rightarrow \quad \frac{dV_T}{V_T} = \frac{d\phi}{1-\phi}$$

from definition of compressibility $\frac{dV_T}{V_T} = -c_r\, d\sigma'$

so that $\quad \frac{d\phi}{1-\phi} = -c_r\, d\sigma'$

using Terzaghi $\quad d\sigma' = d\sigma_T - dp_f$

finally $\quad d\phi = c_r(1-\phi)(dp - d\sigma_T)$

assuming $d\sigma_T = 0$ integrate: $\boxed{\phi = 1 + (1-\phi_0)\, e^{-\alpha(p_f - p_0)}}$

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Terzaghi's principle: $\underset{\text{solid}}{\sigma_T = \sigma'} + \underset{\text{fluid}}{p_f}$

Volumetric strain rate: $\dot{\varepsilon}_{vol} = \frac{1}{V_T}\frac{dV_T}{dt} = \nabla \cdot \underline{v}_s$ constitutive law

from compressibility: $\frac{1}{V_T}\frac{dV_T}{dt} = -c_r\frac{d\sigma'}{dt} = c_r\left(\frac{\partial p_f}{\partial t} - \frac{\partial \sigma_T}{\partial t}\right)$

$$\Rightarrow \quad \boxed{\nabla \cdot \underline{v}_s = c_r\left(\frac{\partial p_f}{\partial t} + \frac{\partial \sigma_T}{\partial t}\right)}$$

substitute into continuity equation together with Darcy's law

$$\nabla\cdot\left(q_r + \underline{v}_s\right) = -\frac{\Delta\rho}{\rho_f\rho_s}\Gamma$$

$$-\nabla\cdot\left(\frac{k}{\mu}\left(\nabla p_f + \rho_f g\hat{z}\right)\right) + c_r\frac{\partial p_f}{\partial t} - c_r\frac{\partial \sigma_T}{\partial t} = -\frac{\Delta\rho}{\rho_f\rho_s}\Gamma$$

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Notes:
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- transient, i.e. time dependent
- fluid pressure, p_f, is the unknown
- σ_T is a known loading rate

porosity change in an elastic rock is only a function of pressure $\rightarrow$ don't need an evolution equation

$$c_r = -\frac{1}{V_T}\frac{dV_T}{d\bar{\sigma}'}\Big|_T \qquad \text{where } V_T = V_f + V_s$$

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