Lecture 10: Cylindrical & spherical shell coords

Logistics: - HW 2 ✓

- HW3 due Thursday!

- HW4 → variable coefficients

Last time: - Neumann BC's & fluxes

- Example: Linear steady confined aquifer with precip & pole recharge

$$\Pi = \frac{Q_i}{Q_p}$$

- Neumann BC: $f_n = q_b A/V$ source/sink

solve_Lbvp(L, fs+fn, ... ⇒ $\underline{\underline{L}} * \underline{h} = \underline{f_s} + \underline{f_n} = f$

- Fluxes: - $q = -\underline{\underline{Kd}} * \underline{\underline{G}} * \underline{h}$ interior

$q_b = \pm r_b V/A$ boundary

⇒ conserves mass exactly

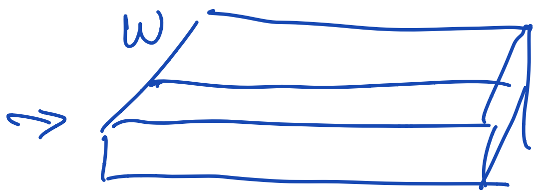
Today: - New coordinate systems

Cylindrical coordinates

Problem with poles recharge   Clifford 1993

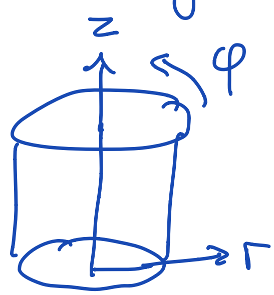
$Q_i = 0.01 - 2.3 \ km^3/yr \qquad \frac{L^3}{T}$

$q_b = \frac{Q_i}{A} = q_i \ \frac{L^3}{L^2 T} = \frac{L}{T} \qquad A = b \ \underset{?}{W}$



$q_b \sim \frac{1}{W}$

$\Rightarrow$ only way to make Q information meaningful is to go to cylindrical coordinates.



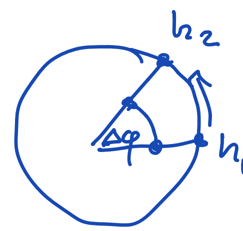
One of the reasons for $\nabla \quad \nabla\cdot \quad \nabla\times$ notation is that it hides the dimension and the coordinate system!

$-\nabla\cdot[K \nabla h] = f_s$ is true in any dim. & coord. syst.

What are operators in cyl. coord:

$$\nabla h = \frac{\partial h}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial h}{\partial \varphi}\hat{\varphi} + \frac{\partial h}{\partial z}\hat{z}$$

$$\nabla \cdot q = \frac{1}{r}\frac{\partial}{\partial r}(r q_r) + \frac{1}{r}\frac{\partial q_\varphi}{\partial \varphi} + \frac{\partial q_z}{\partial z}$$



$$\frac{1}{r}\frac{h_1 - h_2}{\Delta\varphi}$$

$$q = (q_r \; q_\varphi \; q_z)$$

To maintain 1D model we assume

soln is const in φ-direction.

$$\nabla h = \frac{dh}{dx} \qquad \boxed{r = x}$$

$$\nabla \cdot q = \frac{1}{x}\frac{d}{dx}(x\, q)$$

$\Rightarrow$ discrete gradient stays the same $\underline{\underline{G}}$

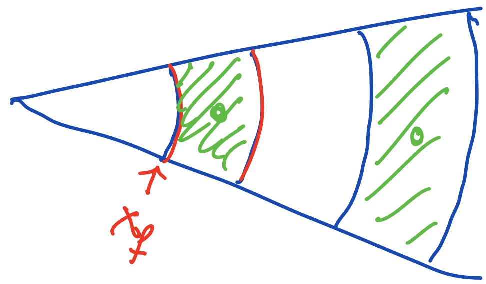
modify discrete divergence $\underline{\underline{D}}$

Geometric interpretation of $x = r$ terms

$$-\frac{1}{x}\frac{d}{dx}\left(x\, b\, K \frac{dh}{dx}\right) = f_s$$

multiply by x

$$-\frac{d}{dx}\left(\underbrace{xb}_{A}\; k \underbrace{\frac{dh}{dx}}_{-q}\right) = x f_s$$



x_f

x inside the div. accounts for increasing A with rad.
$\Rightarrow x = x_f$ faces

x outside divergence accounts for the increase in cell volume as radius increases
$x = x_c$ cell centers.

$\Rightarrow$ Live Script

Example problem: Steady confined aquifer
polar recharge in cylind. coord.

PDE: $-\frac{1}{r}\frac{d}{dr}\left[r\, b\, K \frac{dh}{dr}\right] = 0 \qquad r \in [r_p, \ell]$

BC: $q_i = \frac{Q_i}{A} = -K \frac{dh}{dr}\Big|_{r_p} \qquad -\frac{dh}{dr}\Big|_{r_p} = \frac{q_i}{K}$ dichotomy

$$h(\ell) = h_0$$

Non-dim.: $r' = \frac{r}{\ell}$ $\qquad h' = \frac{h - h_0}{h_c}$

subst:

PDE: $-\frac{d}{dr'}\left[r' \frac{dh'}{dr'}\right] = 0 \qquad r' \in [\rho, 1] \quad \rho = \frac{r_p}{\ell}$

BC: $q' = -\left.\frac{dh'}{dr'}\right|_{\rho} = 1 \qquad h'(1) = 0$

Analytic soln: $h' = -\rho \log(r') \qquad h_c = \frac{q_i \ell}{K}$

$$q' = -\frac{dh'}{dr'} = \frac{\rho}{r'}$$

Dim: $h = h_0 - \frac{q_i \cancel{\ell}}{K} \frac{r_p}{\cancel{\ell}} \log\left(\frac{r}{\ell}\right)$

Spherical shell coordinates

Steady confined aquifer with precipitation on a spherical shell.

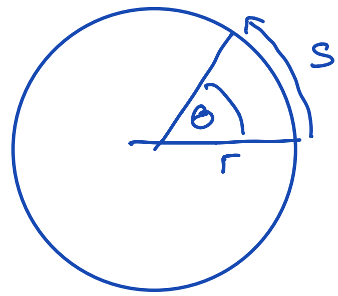
PDE: $-\frac{1}{R\sin\theta}\frac{d}{d\theta}\left[\sin\theta\, bk\, \frac{1}{R}\frac{dh}{d\theta}\right] = q_p$ on $\theta \in [0, \theta_b]$

BC: $q = -k\frac{dh}{d\theta}\Big|_0 = 0 \qquad h(\theta_b) = h_0$

Dimensionless equations

What about θ? $\quad \theta = \frac{s}{r}$

$\theta \in [0, \pi]$



Only need to scale h: $\quad h' = \frac{h - h_0}{h_c}$

substitute

PDE: $\quad -\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\,\frac{dh'}{d\theta}\right] = \frac{q_b R^2}{k b h_c} = 1 \qquad \boxed{h_c = \frac{q_b R^2}{k b}}$

BC: $\quad q' = -\frac{dh'}{d\theta} = 0 \qquad h'(\theta_b) = 0$

Integrate $\quad -\frac{d}{d\theta}\left[\sin\theta\,\frac{dh'}{d\theta}\right] = \sin\theta$

$$\sin\theta \frac{dh'}{d\theta} = \cos\theta + c_1$$

Neumann B: $c_1 = -1$

$$\frac{dh'}{d\theta} = \frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta} = \cot\theta - \csc\theta$$

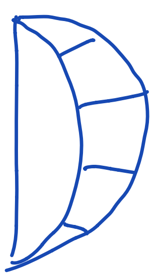
Integrate: $h' = \int \cot\theta - \cos\theta \, d\theta$

$$= \log(\cos\theta + 1) + c_2$$

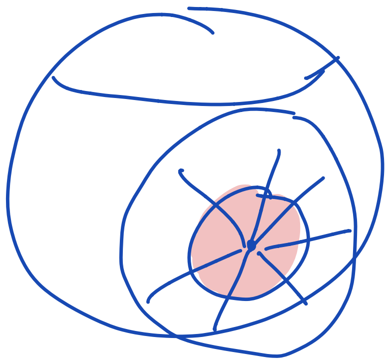
BC at $\theta_b \Rightarrow \quad c_2 = -\log(\cos\theta_b + 1)$

Analytic solution: $h' = \log\left(\frac{\cos\theta + 1}{\cos\theta_b + 1}\right)$

$$q' = -\frac{dh'}{d\theta} = \csc\theta - \cot\theta$$



Now we have meaningful geometry for very large scale problems.



Sperical cap around hellas basin to look at filling of hellas.

Next steps:

Steady ~~Linear~~ confined aquifer

spherical cap

transient (easy)

unconfined (challenging)

non-linear