Lecture 11: Slightly compressible flow

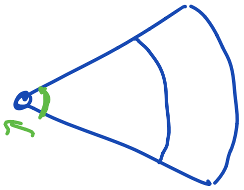
Logistics: - HW 3 8/9

- HW 4 almost done - $\underline{\underline{D}}*\underline{\underline{Kd}}*\underline{\underline{G}}$

you can get started we have covered everything

Last time: Cylindrical & Spherical shell coordinates

Example: Polar recharge in cylind. coords.



$$\nabla h = \frac{dh}{dr} \qquad \nabla \cdot q = \frac{1}{r}\frac{d}{dr}(r\, q)$$

$$\underline{\underline{D}} = \underline{\underline{Rinv}} * \underline{\underline{D}} * \underline{\underline{R}}$$

$\nearrow$ diagonal with 1/$\underline{xc}$

$\uparrow$ stand. div

$\nwarrow$ diagonal with $\underline{xf}$

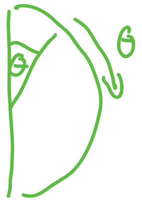
$\Rightarrow$ solution has strong bnd layer

$$h \sim -\underline{\log r} \qquad q \sim \underline{\frac{1}{r}}$$

need mass conservation for good answer

Example: Aquifer with precip on spher. shell

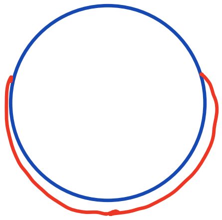
$$\nabla h = \frac{1}{R}\frac{dh}{d\theta} \qquad \nabla\cdot q = \frac{1}{R\sin\theta}\frac{d}{d\theta}(\sin\theta\, q)$$



⇒ simply modify existing coords

Analytic solution

$$h = \log\left(\frac{\cos\theta+1}{\cos\theta_b+1}\right)$$

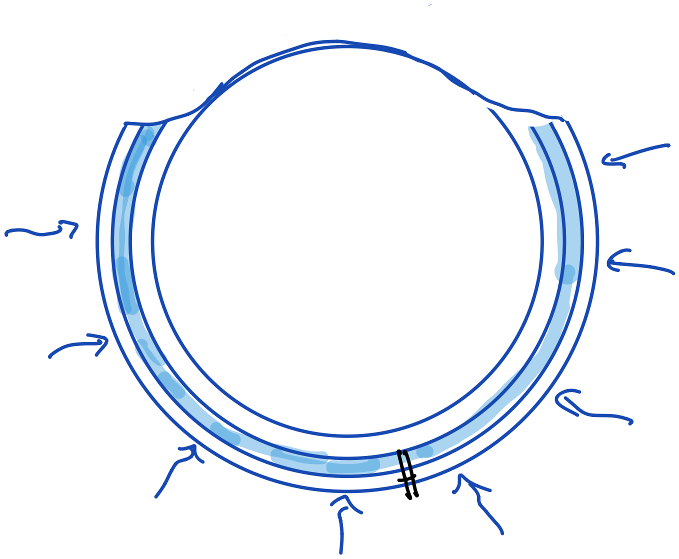


Today: Slightly compressible flow

⇒ transient behavior

Example: Drainage of linear aquifer

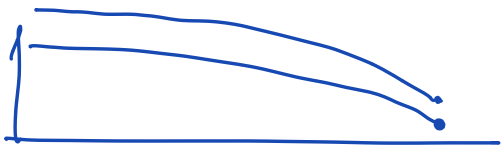
⇒ Self-similarity



Slightly compressible flow

So far incompressible flow, $\rho_f = \text{const}$ $\phi = \text{const}$

$$-\nabla\cdot K\nabla h = f_s \qquad \text{steady eqn}$$

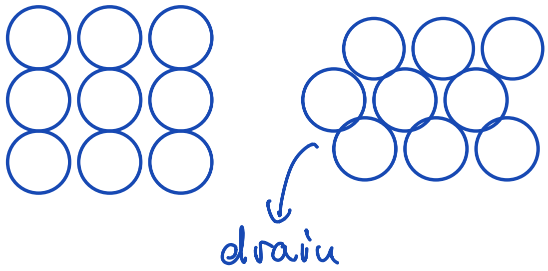


The only temporal change is due to BCs

$\Rightarrow$ succession of steady states.

Transient behavior $\frac{\partial}{\partial t}(\rho\phi) \neq 0$ and hence if either $\rho \neq \text{cost}$ or $\phi \neq \text{cost}$ (with time)

In general fluids are more compressible than solids. But in GW systems the pressures are to low for either fluid or solid to compress. Aquifer compressibility arises from the <u>two-phase nature</u> of the system and is called <u>consolidation</u> or <u>compaction</u>.



During compaction the grains re-arrange to reduce porosity

$\Rightarrow$ gives rise to effective compressibility

Balance of fluid and solid mass

Fluid: $\frac{\partial}{\partial t}(\cancel{\rho_f}\phi) + \nabla\cdot[\phi\,\underline{v}_f\,\cancel{\rho_f}] = 0$

Solid: $\frac{\partial}{\partial t}\left(\cancel{\rho_s}(1-\phi)\right) + \nabla\cdot[(1-\phi)\,\underline{v}_s\,\cancel{\rho_s}] = 0$

Assume $\rho_s = \text{const}$ $\rho_f = \text{const}.$

Darcy's law: $\underline{q} = \phi\,(\underline{v}_f - \underline{v}_s) = -K\nabla h$

relative flux

Add balance equations:

$$\cancel{\frac{\partial}{\partial t}(\cancel{\phi} + 1 - \cancel{\phi})} + \nabla\cdot[\phi\,\underline{v}_f + (1-\phi)\,\underline{v}_s] = 0$$

$$\boxed{\begin{aligned}\nabla\cdot[\phi\,\underline{v}_f + (1-\phi)\,\underline{v}_s] &= 0\\ \nabla\cdot[\underline{q} + \underline{v}_s] &= 0\end{aligned}}$$

Continuity Equation

$\nabla \cdot q$ → Darcy's law

$\nabla \cdot v_s$ → ? rock mechanics

Flow in an elastic rock

Bulk rock compressibility: $c_r = -\frac{1}{V_T}\frac{dV_T}{\underline{d\sigma'}}\Big|_{Temp} \sim 10^{-8}\,\frac{1}{Pa}$

$V_T = V_f + V_R$

σ' = effective stress (stress/weight on rock)

Terzaghi's principle: $\boxed{\sigma_T = \sigma' + p}$

total stress

pore pressure

Volumetric strain rate: $\dot{\varepsilon}_{vol} = \frac{1}{V_T}\frac{dV_T}{\underline{dt}} = \underline{\nabla \cdot \underline{v}_s}$

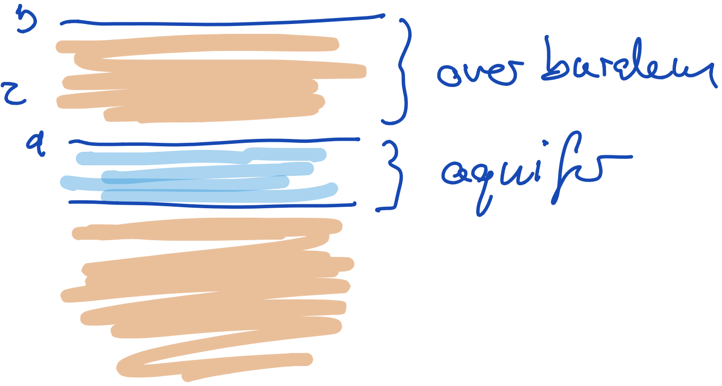
from def. of comp. $\frac{1}{V_T}\,dV_T = -\,c_r\,d\sigma'$

$$\nabla \cdot \underline{v}_s = \frac{1}{V_T}\frac{dV_T}{dt} = -\,c_r\frac{d\sigma'}{dt} = c_r\left(\frac{dp}{dt} - \frac{d\sigma_T}{dt}\right)$$

using Terzaghi $\sigma' = \sigma_T - p$

$$\boxed{\nabla\cdot \underline{v}_s = c_r\left(\frac{dp}{dt} - \frac{d\sigma_T}{dt}\right)}$$ constitutive equ for elastic rock.

$$\sigma_T = \int_a^b \rho_s g\, dz$$



~~$\sigma_T = w$~~

σ_T weight of over burden

Convert p. to h: $$h = z + \frac{p - p_0}{\rho_f g}$$

$$\frac{dh}{dt} = \frac{1}{\rho_f g}\frac{dp}{dt}$$

$$\nabla\cdot v_s = c_r\left(\rho_f g \frac{dh}{dt} - \frac{d\sigma_T}{dt}\right)$$

substitute into continuity with Darcy's law

$$\nabla\cdot \underline{v}_s + \nabla\cdot \underline{q} = 0$$

$$c_r \rho_f g \frac{dh}{dt} - \nabla\cdot K \nabla h = c_r \frac{d\sigma_T}{dt}$$

Specific storage: $\boxed{S_s = c_r \rho_f g}$ $\quad \frac{L\cancel{T^2}}{\cancel{M}}\,\frac{\cancel{M}}{L^3}\,\frac{L}{\cancel{T^2}} = \frac{1}{L}$

$Pa = \frac{F}{A} = \frac{ML}{T^2L^2} = \frac{M}{LT^2}$

Discharge: Q $\frac{L^3}{T}$

spec. discharge: $q = \frac{Q}{A}$

$\frac{L}{T} = \frac{L^3}{L^2T}$

Physical interpretation:

S_s is the volume of fluid stored/released per unit volume of rock per unit decrease in head. $\frac{L^3}{L^3}\frac{1}{L} = \frac{1}{L}$

$c_r \sim 10^{-8} \frac{1}{Pa}$ $\rho_f \sim 10^3 \frac{kg}{m^3}$ $g \sim 1 \frac{m}{s^2}$

$$S_s \sim 10^{-8+3+1} \frac{1}{m} = 10^{-4} \frac{1}{m}$$

For 1m of head drop ~ 100 ml of water are released from the rock due to consolidation.

Note: Here we assume compaction is reversible, which is only partially true.

Transient groundwater flow equation (confined)

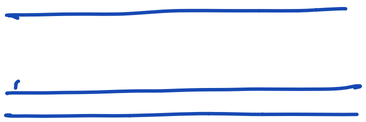
$$\boxed{S_s \frac{\partial h}{\partial t} - \nabla \cdot K \nabla h = c_r \frac{d\sigma_T}{dt}}$$

$\underline{\text{Linear}}$ equation

Note: $q = \phi(v_f - \underset{\uparrow}{v_s}) = -K\nabla h$

dropped this term $\underline{v}_s \ll 1 \sim \frac{cm}{yr}$

Typically $\frac{d\sigma_T}{dt} = 0$



$\frac{d\sigma_T}{dt} > 0$