Lecture 12: Time integration

Logistics: – HW4 due Th

– HW5 over spring break → spherical shell

Last time: slightly compressible flows

- Two-phase continuity: $\nabla \cdot [\underline{q}_r + \underline{v}_s] = 0$

2 constitutive relations:

1) Darcy: $q_r = -K \nabla h$

2) Elasticity: $\nabla \cdot \underline{v}_s = c_r \left(\rho_f g \frac{\partial h}{\partial t} - \frac{d\sigma_T}{dt} \right)$

New material parameter:

Specific storage: $S_s = \rho_f g \, c_r$

Transient ground water flow equation

$$\boxed{S_s \frac{\partial h}{\partial t} - \nabla \cdot [K \nabla h] = f_s}$$

linear

Today: - Discretizing the time derivative

- Theta method

- Numerical stability

Time integration

Transient linear PDE

$$\underbrace{S_s \frac{\partial h}{\partial t}}_{\text{new}} - \underbrace{\nabla \cdot K \nabla h}_{\underline{\underline{L}}\,\underline{h} = (-\underline{\underline{D}} * \underline{\underline{Kd}} * \underline{\underline{G}}) * \underline{h}} = f_s \; (+ f_n)$$

Mass matrix: $\underline{\underline{M}} = S_s \underline{\underline{I}}$ homoge

$= \text{spdiags}(\underline{S_s}, Nx, Nx)$

Finite difference in time:

$$\frac{\partial h}{\partial t} \approx \frac{h^{n+1} - h^n}{\Delta t} \qquad \Delta t = t^{n+1} - t^n$$

substitute:

$$\underline{\underline{M}} (\underline{h}^{n+1} - \underline{h}^n) + \Delta t \, \underline{\underline{L}} \, \underline{h}^{?} = \Delta t (\underline{f_s} + \underline{f_n})$$

Theta Method

$$\underline{h}^{\theta} = \theta \, \underline{h}^n + (1-\theta) \, \underline{h}^{n+1}$$

⇒ combines the 3 most common method

substitute

$$\underline{\underline{M}} (\underline{h}^{n+1} - \underline{h}^n) + \Delta t \, \underline{\underline{L}} \left[\theta \, \underline{h}^n + (1-\theta) \, \underline{h}^{n+1} \right] = \Delta t (\underline{f_s} + \underline{f_n})$$

collect $\underline{h}^{n+1}$ on lhs

$$\underbrace{\left[\underline{\underline{M}} + \Delta t(1-\theta)\underline{\underline{L}}\right]}_{\underline{\underline{IM}}} \underline{h}^{n+1} = \Delta t(\underline{f_s} + \underline{f_n}) + \underbrace{\left[\underline{\underline{M}} - \Delta t\,\theta\,\underline{\underline{L}}\right]}_{\underline{\underline{Ex}}} \underline{h}^{n}$$

Linear system for a time step

$$\boxed{\underline{\underline{IM}}\,\underline{h}^{n+1} = \Delta t(\underline{f_s} + \underline{f_n}) + \underline{\underline{EX}}\,\underline{h}^{n}}$$

$$\underline{\underline{IM}} = \underline{\underline{M}} + \Delta t\,(1-\theta)\,\underline{\underline{L}}$$

$$\underline{\underline{EX}} = \underline{\underline{M}} - \Delta t\,\theta\,\underline{\underline{L}}$$

This time integration scheme applies to every linear transient problem, because $\underline{\underline{M}}$ and $\underline{\underline{L}}$ hide the details of PDE

Properties of theta method

$\theta = 1$: Forward Euler Method

$$\underline{\underline{IM}} = \underline{\underline{M}} \qquad \text{(diagonal)}$$

$$\Rightarrow \quad \underline{h}^{n+1} = \underset{\substack{\uparrow \\ \text{trivial}}}{\underline{\underline{M}}^{-1}} \left(\Delta t\,\underline{f_s} + \underline{\underline{EX}}\,\underline{h}^{n}\right)$$

- explicit method
  - → only matrix-vector multiply ($\underline{\underline{EX}} * \underline{h}^n$)
  - → cheap
- conditionally stable $\Delta t \leq \frac{\Delta x^2}{2 D_{hyd}}$

  $D_{hyd} = K / S_s \quad [L^2/T]$
- first order accurate

$\theta = 0$: Backward Euler Method

$\underline{\underline{EX}} = \underline{\underline{M}}$

$\underline{\underline{IM}}\ \underline{h}^{n+1} = \Delta t(\underline{f_s} + f_u) + \underline{\underline{EX}}\ \underline{h}^n$

- implicit method → solve linear system at each time step
- unconditionally stable
- first-order accurate

⇒ choose Δt by accuracy considerations

$\theta = \frac{1}{2}$: Crank-Nicholson method

- implicit method → solve linear sys.
- unconditionally stable.
- second order accurate
- oscillation limit → see Live script

Backward Euler is safe choice

Amplification matrix

$$\underline{\underline{IM}}\ \underline{h}^{n+1} = \underline{\underline{EX}}\ \underline{h}^{n}$$

$$\underline{h}^{n+1} = \underbrace{\underline{\underline{IM}}^{-1}\underline{\underline{EX}}}_{\underline{\underline{A}}}\ \underline{h}^{n}$$

$$\underline{h}^{n+1} = \underline{\underline{A}}\ \underline{h}^{n} \qquad \underline{\underline{A}} = \underline{\underline{IM}}^{-1}\ \underline{\underline{EX}}$$

forming $\underline{\underline{A}}$ for an implicit method is very expensive and not practical, but it helps understand stability.

$\underline{h}^1 = \underline{\underline{A}}\ \underline{h}^0$

initial

$\underline{h}^2 = \underline{\underline{A}}\ \underline{h}^1 = \underline{\underline{A}} * \underline{\underline{A}}\ \underline{h}^0$

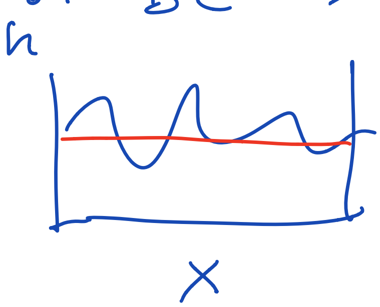
$\underline{h}^n = \underline{\underline{A}}^n\, \underline{h}^0$

$\underline{\underline{A}}^n = \underbrace{\underline{\underline{V}}^T\, \underline{\underline{\Lambda}}^n\, \underline{\underline{V}}}_{\text{spectral decomp.}}$

$\underline{\underline{V}}$ = matrix of eigenv.

$\underline{\underline{\Lambda}}$ = diag. matrix of eigenvalues

→ solving a diff. problem in absence of exitation from source or BC → decay



Only happens if the eigenvalues of $\underline{\underline{A}}$ are all $|\lambda| < 1$

$$h = \sum a_n \sin\left(\frac{2\pi n x}{L}\right) e^{-\lambda_n D t}$$