Lecture 13: Self-similar diffusion

Logistics: - HW 4 is due today

- HW 5 is available - spherical cap geom

Last time: - time integration

$$S_s \frac{\partial h}{\partial t} - \underbrace{\nabla \cdot [K \nabla h]} = f_s$$

$$\underline{\underline{M}} \frac{h^{n+1} - h^n}{\Delta t} + \underline{\underline{L}}\, \underline{h}^{?} = \underline{f_s}$$

- theta method: $\underline{h} = \theta h^n + (1-\theta) h^{n+1}$

$$\boxed{\underline{\underline{IM}}\, \underline{h}^{n+1} = \Delta t\, f_s + \underline{\underline{EX}}\, \underline{h}^n}$$

$$\underline{\underline{IM}} = \underline{\underline{M}} + \Delta t (1-\theta) \underline{\underline{L}} \qquad \underline{\underline{EX}} = \underline{\underline{M}} + \Delta t\, \theta\, \underline{\underline{L}}$$

$\theta = 1$: Forward Euler. $\Delta t \leq \frac{\Delta x^2}{2 D_{hyd}}$ $D_{hyd} = K/S_s$

$\theta = 0$: Backward Euler $\rightarrow$ safest

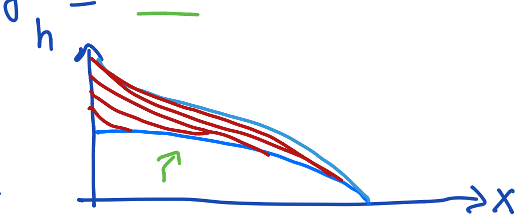
$\theta = \frac{1}{2}$: Crank-Nicholson $\rightarrow$ higher order.

- Amplification matrix: $\boxed{\underline{\underline{A}} = \underline{\underline{IM}}^{-1} \underline{\underline{EX}}}$ $\underline{h}^n = \underline{\underline{A}}^n \underline{h}^0$

$\rightarrow$ stable if eigs of $\underline{\underline{A}} \leq 1$

- Example:

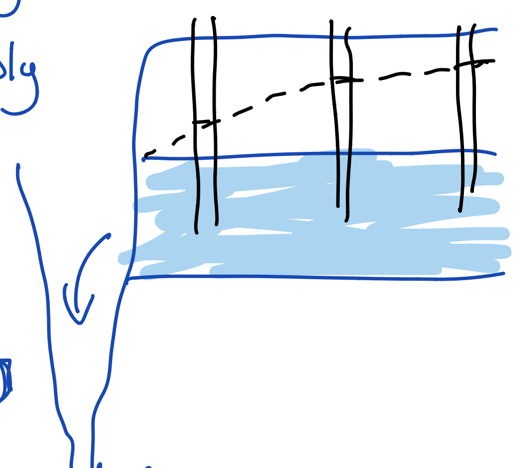
Transient recharge



Today: Classic self similar probems
in linear diffusion
- drainage from lin. conf. aquifer
- recharge of lin conf. aquifer

Example 1: Aquifer draining into Valles Mariner's

Imagine an instanteneously formed crack.



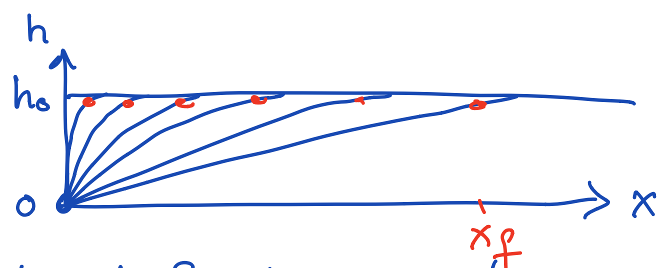
PDE: $\frac{\partial h}{\partial t} - \nabla \cdot D_{hyd} \nabla h = 0$

$x \in [0, \infty]$

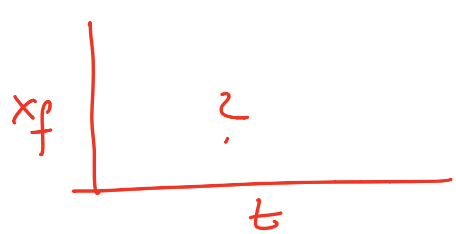
BC: $h(0,t) = 0$ h is bounded as $x \to \infty$

IC: $h(x,0) = h_0$

$D_{hyd} = K/S_s$



How fast does the head front propagate



Scaling the problem

There is no external length scale!

Note $\sqrt{D_{hyd}\, t}$ $\sqrt{\frac{L^2}{T} T} = L$ is a length scale!

internal length scale = diffusive length scale

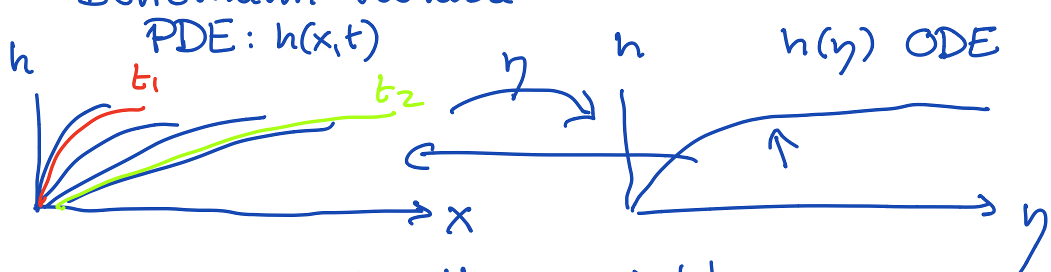
What about $x' = \frac{x}{\sqrt{Dt}}$?

scaling one independent variable x with

another indep. variable t

$\Rightarrow$ new independent variable $\boxed{\eta = \frac{x}{\sqrt{4Dt}}}$

"Boltzmann variable"



η is a similarity variable

What is the self-similar ODE?

First we scale $h' = \frac{h}{h_0}$ $\rightarrow$ IC $h'(x,0) = 1$

Solution: $h'(x,t) = \Pi(\eta(x,t))$

Transform the derivatives:

$$\frac{\partial h'}{\partial t} = \frac{\partial \Pi}{\partial t} = \frac{d\Pi}{d\eta}\frac{\partial \eta}{\partial t} \qquad \frac{\partial h'}{\partial x} = \frac{\partial \Pi}{\partial x} = \frac{d\Pi}{d\eta}\frac{\partial \eta}{\partial x}$$

$$\eta = \frac{x}{\sqrt{4Dt}} : \quad \frac{\partial \eta}{\partial x} = \frac{1}{\sqrt{4Dt}} \neq f(x) \qquad \frac{\partial \eta}{\partial t} = \frac{-\eta}{2t}$$

$$\frac{\partial^2 h'}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{d\Pi}{d\eta}\frac{\partial \eta}{\partial x}\right) = \frac{\partial \eta}{\partial x}\frac{d^2\Pi}{d\eta^2}\frac{\partial \eta}{\partial x} = \left(\frac{\partial \eta}{\partial x}\right)^2 \frac{d^2\Pi}{d\eta^2}$$

$$= \frac{1}{4Dt}\frac{d^2\Pi}{d\eta^2}$$

$$D = D_{hyd} = K/S_s$$

Substitute into PDE:

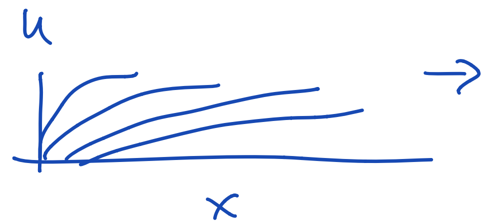
$$\frac{\partial h'}{\partial t} - D\frac{\partial^2 h'}{\partial x^2} = 0$$

$$2\frac{d\Pi}{d\eta}\left(-\frac{\eta}{\cancel{2}\cancel{t}}\right) - \frac{\cancel{D}}{\cancel{4}\cancel{D}\cancel{t}}\frac{d^2\Pi}{d\eta^2} = 0$$

ODE: $$\boxed{\frac{d^2\Pi}{d\eta^2} + 2\eta\frac{d\Pi}{d\eta} = 0 \qquad \eta \in [0,\infty)}$$

BC: $$\boxed{\Pi(\eta=0) = 0 \qquad \lim_{\eta\to\infty}\Pi = 1 \qquad (IC)}$$

for finite t $\quad t \neq 0 \quad t \neq \infty$



$$\eta = \frac{x}{\sqrt{4Dt}}$$

$$h(x, t=0)$$

Solve ODE:

1) substitute: $u = \frac{d\Pi}{d\eta} \Rightarrow \frac{du}{d\eta} + 2\eta u = 0$

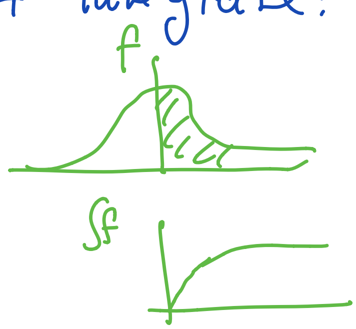
2) sep. variables: $\frac{du}{u} = -2\eta \, d\eta$

$$\log u = -\eta^2 + a$$

$$u = c\, e^{-\eta^2}$$

3) resubstitute: $\frac{d\Pi}{d\eta} = c\, e^{-\eta^2}$

4 integrate: $\Pi = c \int_0^{\eta} \underbrace{e^{-\hat{\eta}^2}}_{f} d\hat{\eta}$ $\quad$ $\hat{\eta}$ dummy variable

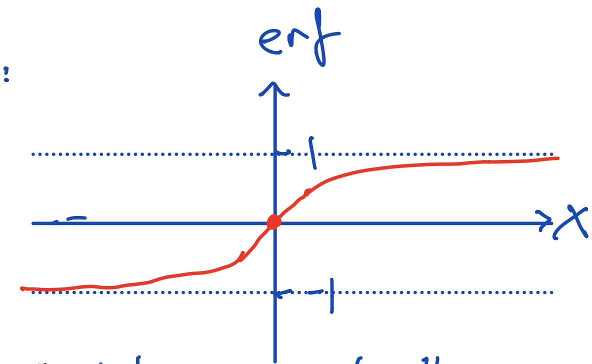


integral does not have analytic solution!

$\Rightarrow$ give integral name & move on

5) Identify error function:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\, dz$$



Properties of error function:

- $\mathrm{erf}(-x) = -\mathrm{erf}(x)$ "point symmetry"
- $\mathrm{erf}(0) = 0$
- $\mathrm{erf}(x) \approx x$ $\quad |x| \ll 1$
- $\lim_{x \to \infty} \mathrm{erf}(x) = 1$

Therefore: $\Pi(\eta) = c \frac{\sqrt{\pi}}{2} \mathrm{erf}(\eta)$

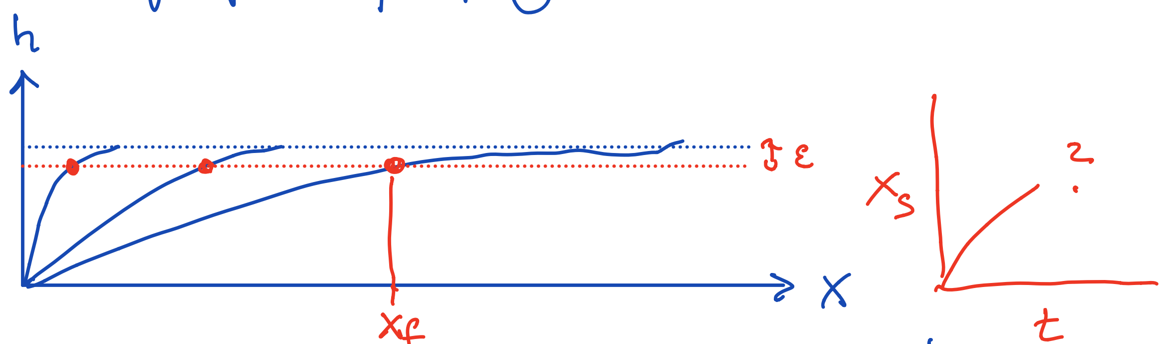
BC $\lim_{\eta \to \infty} \Pi(\eta) = \underset{\uparrow \atop c = \frac{2}{\sqrt{\pi}}}{c} \frac{\sqrt{\pi}}{2} \underbrace{\mathrm{erf}(\eta)}_{1} = 1$

$\Rightarrow$ $\boxed{\Pi(\eta) = \mathrm{erf}(\eta)}$ Self similar solution

6. Resubstitute: $h = h_0\, h'$ $\qquad \eta = \frac{x}{\sqrt{4Dt}}$

$$\boxed{h(x,t) = h_0\, \mathrm{erf}\left(\frac{x}{\sqrt{4Dt}}\right)}$$

Speed of front propagation



Note: Infinitesimal changes in h propagate instantaneously all the way to infinity

Finite changes propagate with finite speed

The front is defined as location, x_f, where h has changed by εh_0

We are looking for:

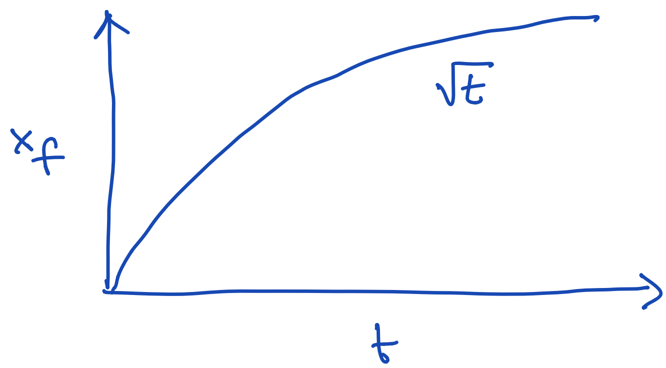
$$h(x_f, t) = h_0 - \varepsilon h_0 = h_0 (1-\varepsilon)$$

subst into analytic soln.

$$\cancel{h_0}(1-\varepsilon) = \cancel{h_0}\, \mathrm{erf}\left(\frac{x_f}{\sqrt{4Dt}}\right)$$

$$\frac{x_f}{\sqrt{4Dt}} = \mathrm{erf}^{-1}(1-\varepsilon) = \alpha(\varepsilon)$$

$$\Rightarrow \quad \boxed{x_f = \alpha(\varepsilon)\sqrt{4Dt}} \qquad x_f \sim \sqrt{t}$$



$$v_f = \frac{dx_f}{dt} \sim \frac{1}{\sqrt{t}}$$

Why: $\quad x \sim t^{\frac{1}{2}} \qquad x^2 \sim t$

$$\frac{dy}{dt} = \frac{d^2y}{dx^2}$$