Lecture 14: Unconfined flow

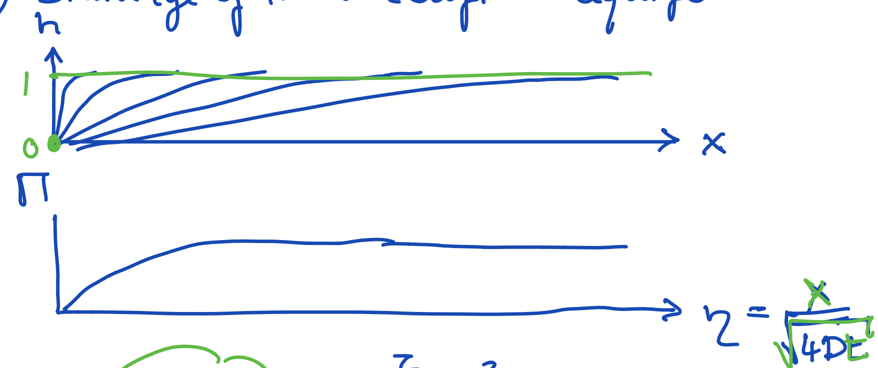
Logistics: - HW5 is due Thursday

- please make use of office hrs!

- great people are helping each other on piazza!

Last time: - classic self-similar diffusion solutions

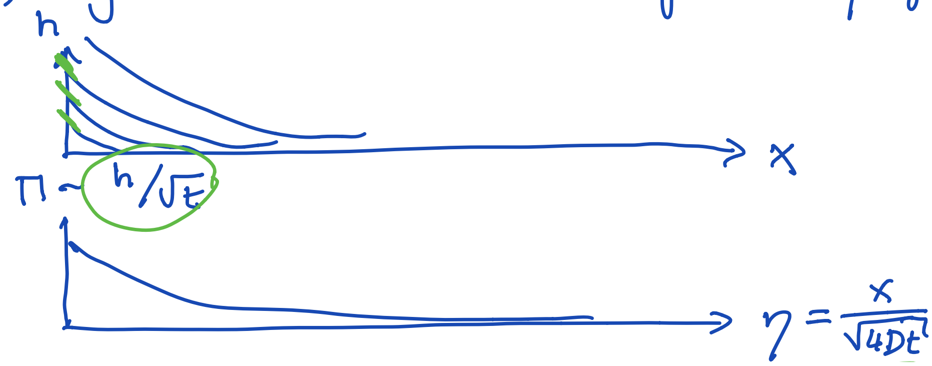
I) Drainage of linear confined aquifer



$\rightarrow \mathrm{erf}(z) = \frac{2}{\pi}\int_0^z e^{-x^2}dx$

Front propagates $\boxed{x \sim \sqrt{t}}$

II) Injection into linear confined aquifer



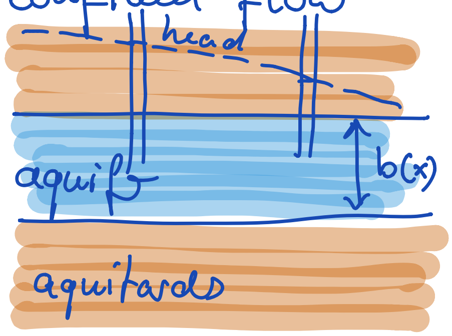
Today: - Intro to unconfined flow

- Newton-Raphson method

- Jacobian matrix for PDE problems

Unconfined flow

confined flow

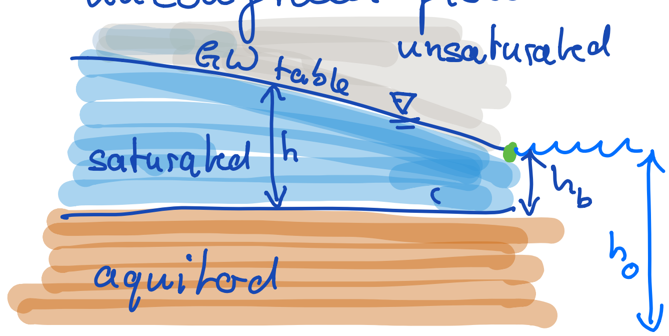


steady flow

$$-\nabla\cdot[K b \nabla h] = f_s$$

linear in h

unconfined flow



steady flow

$$-\nabla\cdot[K h \nabla h] = f_s$$

non-linear in h

Consider discretizing this

continuous pde: $\quad -\nabla\cdot[K h \nabla h] = f_s$

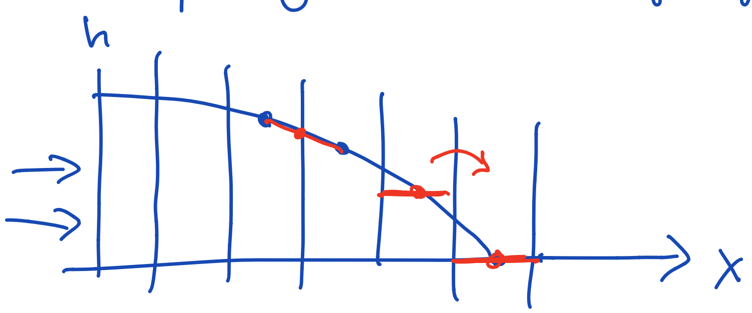
discrete sys.: $\quad -\underline{\underline{D}}\cdot[\underline{\underline{K_d}}\ ?\ \underline{\underline{G}}\,\underline{h}] = \underline{f_s}$

h is just another function that multiplies

∇h $\rightarrow$ treat is similarly to K

$\underline{\underline{Kd}} = \text{spdiags}(\text{Kmean}, 0, Nf, Nf)$ harmonic ave.

$\underline{\underline{Hd}} = \text{spdiags}(\text{hmean}, 0, Nf, Nf)$ arithmetic ave.



h is smooth function

linear interp. to bnd $\rightarrow$ arithm.

harmonic ave prevents flow into initially dry cells

residual : $\underline{r}(\underline{h}) = \underline{\underline{D}}[\underline{\underline{Kd}}\,\underline{\underline{Hd}}(\underline{h})\,\underline{\underline{G}}\,\underline{h}] + \underline{f_s} = 0$

$$\Rightarrow \boxed{-\underline{\underline{D}}[\underline{\underline{Kd}}\,\underline{\underline{Hd}}(\underline{h})\,\underline{\underline{G}}\,\underline{h}] = f_s}$$

$$\underline{\underline{L}}\,\underline{h} \neq f_s$$

because the problem is non-linear we cannot form $\underline{\underline{L}}$ $\Rightarrow$ non-linear algebraic sys.

Example: Linear unconfined aquifer with precip.

PDE: $-\frac{d}{dx}\left(K h \frac{dh}{dx}\right) = q_p \qquad x \in [0, \ell]$

BC: $\left.\frac{dh}{dx}\right|_0 = 0 \qquad h(\ell) = h_b$

Scale the problem: $h' = \frac{h \cancel{- h_b}}{h_c} \qquad x' = \frac{x}{\ell}$

$$-\frac{K h_c^2}{\ell^2} \frac{d}{dx'}\left(h' \frac{dh'}{dx'}\right) = q_p \qquad x' \in [0, 1]$$

$$\frac{d}{dx'}\left(h' \frac{dh'}{dx'}\right) = \frac{q_p \ell^2}{K h_c^2} = 1 \quad \Rightarrow \quad \boxed{h_c^u = \sqrt{\frac{q_p \ell^2}{K}}}$$

confined $\qquad \frac{q_p \ell^2}{K b h_c} = 1 \qquad h_c^c = \frac{q_p \ell^2}{K b}$

Dimensionless problem:

PDE: $-\frac{d}{dx'}\left[h' \frac{dh'}{dx'}\right] = 1 \qquad x' \in [0, 1]$

BC: $\left.\frac{dh'}{dx'}\right|_0 = 0 \qquad h'(1) = \frac{h_b}{h_c} = \boxed{\frac{h_b}{\ell}\sqrt{\frac{K}{q_p}} = \Pi}$

Note: Unconfined problem has a dim. parameter, unlike confined equivalent.

Here $h_b \geq 0$ is not sealevel, but the elevation of sealevel above the base of the aquifer (permeable region of crust)

Integrate: $-h\dfrac{dh}{dx} = x + c_1$

NB $x=0$: $-h\,0 = 0 + c_1 \Rightarrow c_1 = 0$

$-h\dfrac{dh}{dx} = x$ $\qquad q' = -\dfrac{dh'}{dx'}$

Integrate: $-h\,dh = x\,dx$ $\qquad q' = \dfrac{x'}{h'}$

$-\dfrac{h^2}{2} = \dfrac{x^2}{2} + c_2$

DB $x=1$: $-\dfrac{\Pi^2}{\cancel{2}} = \dfrac{1}{\cancel{2}} + c_2 \Rightarrow c_2 = -\dfrac{1}{\cancel{2}} - \dfrac{\Pi^2}{\cancel{2}}$

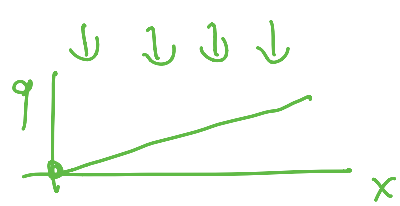
$h^2 = 1 + \Pi^2 - x^2$

$$\boxed{\begin{aligned} h' &= \sqrt{1+\Pi^2-x'^2} \\ q' &= \frac{x'}{h'_b} = \frac{x'}{\sqrt{1+\Pi^2-x'^2}} \end{aligned}}$$

✓

confined

$h' = (1-x^2)$ $\qquad q' = x$

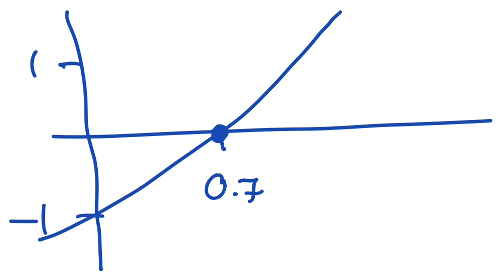


Newton-Raphson method

To find the zero/root of non-linear alg. equn(s)

Suppose a single non linear function

$$r(x) = e^x - 2$$



$e^x = 2 \qquad x = \log 2 \simeq 0.7$

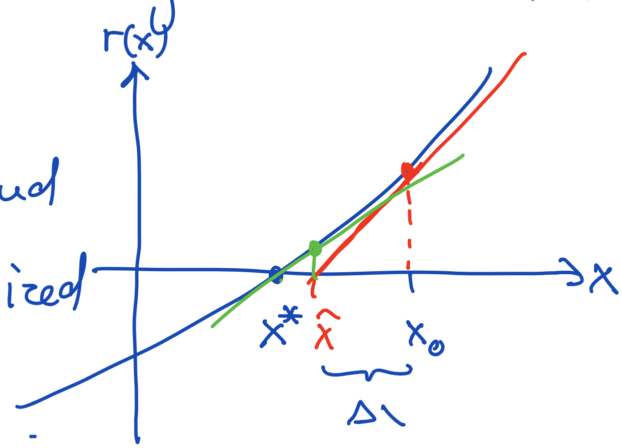
We need to find the root by iteration

$\Rightarrow$ sequence of improving guesses

This requires an initial guess

If the initial guess x_0 is close enough to the root x^*

we can linearize the function, $r(x)$, and find the root of the linearized function.



Taylor series:

$$L_{x_0} r(x) = r(x_0) + \underbrace{(x - x_0)}_{\Delta x} \left.\frac{dr}{dx}\right|_{x_0}$$

Root or $\mathcal{L}_{x_0} r(x) = 0$

$$\Delta x = -r(x_0) \Big/ \left.\frac{dr}{dx}\right|_{x_0}$$

$$\hat{x} = x_0 + \Delta x$$

The Newton-Raphson method turns this into an iterative procedure that converges to the root of $r(x)$ quadratically, If x_0 is the "basin of convergence".

at k-th iteration:

$$\boxed{\begin{aligned} \Delta x^k &= -r(x^k) \Big/ \left.\frac{dr}{dx}\right|_{x^k} \\ x^{k+1} &= x^k + \Delta x^k \end{aligned}}$$

Newton-Raphson for single eqn.

This is in a while loop