Lecture 15: Newton-Raphson for unconfined flow

Logistics: - HW5 due

- HW6 will be posted - transient

Last time: Unconfined flow

$$-\nabla\cdot[b K \nabla h] = f_s \xrightarrow{b \to h} -\nabla\cdot[h K \nabla h] = f_s$$

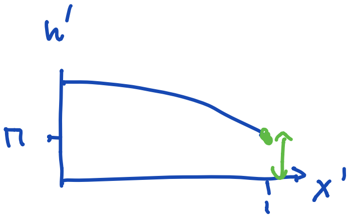
$$h \nabla h$$

residual: $\underline{r}(\underline{h}) = \underline{\underline{D}}[\underline{\underline{H}}(\underline{h})\, \underline{\underline{Kd}}\, \underline{\underline{G}}\, \underline{h}] + \underline{f_s} = 0$ non-linear

$$\uparrow$$

$\underline{\underline{H}}$ = spdiags($\underline{\underline{M}}$ h, 0, Nx, Nx)

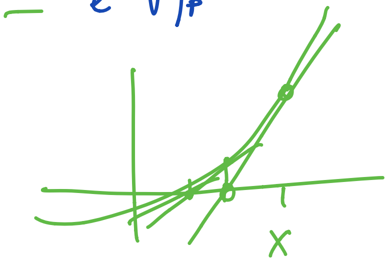
- Example: linear unconfined flow with precip



$$h' = \sqrt{1 + \Pi^2 - x'^2} \qquad q' = \frac{x'}{h'}$$

$$h_c = \sqrt{\frac{q_p\, \ell^2}{K}} \qquad \Pi = \frac{h_b}{\ell}\sqrt{\frac{K}{q_p}}$$

- Newton-Raphson method



$$\Delta x^k = -\, r(x^k) \Big/ \left.\frac{dr}{dx}\right|_{x^k}$$

$$x^{k+1} = x^k + \Delta x^k$$

monitor $|r(x^k)| \le tol$ $\quad |\Delta x^k| \le tol$ $\quad k \le k_{max}$

Today: - Non-linear systems of equations

- Non-linear systems from PDE discretizations

Numerical solution for unconfined flow

Residual of dimensionless system:

$$\underline{r} = -\underline{\underline{D}}\left[\{\underline{\underline{M}}\,\underline{h}\}_f\,\underline{\underline{G}}\,\underline{h}\right] + \underline{f_s}$$

$\underline{\underline{H}}(\underline{h}) = \{\underline{\underline{M}}\,\underline{h}\}_f$ is Nf by Nf ~~a~~ matrix with $\underline{\underline{M}}\,\underline{h}$ on the diagonal

We are looking for root $\underline{h}^*$ such that $\underline{r}(\underline{h}^*) = 0$

Linearize the discrete residual

$\underline{\bar{h}}$ is reference current iterate is known

$\underline{\Delta h}$ is update ~ x_0 not known

$\underline{\Delta h} = \epsilon\,\underline{\hat{h}}$ $\quad\underline{\hat{h}}$ = direction of update

$$L_{\underline{\bar{h}}}\,\underline{r}(\underline{\hat{h}}) = \underline{r}(\underline{\bar{h}}) + \nabla_{\underline{h}}\,\underline{r}(\underline{\bar{h}})\,\Delta h$$

$$= \underline{r}(\underline{\bar{h}}) + \epsilon\,\underbrace{\nabla_{\underline{h}}\,\underline{r}(\underline{\bar{h}})\,\hat{h}}$$

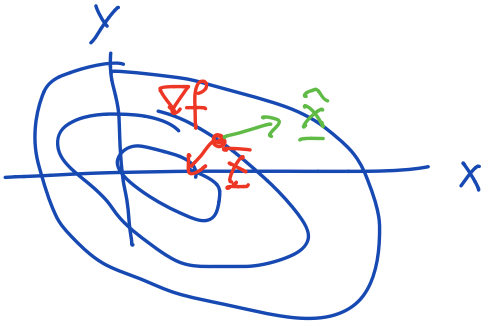
$$\boxed{L_{\underline{\bar{h}}}\,\underline{r}(\underline{\hat{h}}) = \underline{r}(\underline{\bar{h}}) + \epsilon \quad D_{\underline{\hat{h}}}\,\underline{r}(\underline{\bar{h}})}$$

$$L_{x_0} f = f(x_0) + \underset{\epsilon\,\hat{x}}{\Delta x}\,\left.\frac{df}{dx}\right|_{x_0}$$

Directional derivative

$$\boxed{D_{\hat{\underline{h}}}\,\underline{\Gamma}(\bar{\underline{h}}) = \frac{d}{d\epsilon}\,\underline{\underline{\Gamma}(\bar{\underline{h}} + \epsilon\hat{\underline{h}})}\Big|_{\epsilon=0}}$$

Simple example:



$f(\underline{x}) = x + y^2$

$\nabla f = \begin{pmatrix} 1 \\ 2y \end{pmatrix}$

Given some direction $\hat{\underline{x}} = \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}$ and a location $\bar{\underline{x}} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$

a) $D_{\hat{\underline{x}}} f(\bar{\underline{x}}) = \nabla f\big|_{\bar{\underline{x}}} \cdot \hat{\underline{x}} = (1 \quad 2\bar{y}) \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \underline{\boxed{\hat{x} + 2\bar{y}\,\hat{y}}}$

b) $D_{\hat{\underline{x}}} f(\bar{\underline{x}}) = \frac{d}{d\epsilon} f(\bar{\underline{x}} + \epsilon\hat{\underline{x}})\Big|_{\epsilon=0}$

$$\frac{d}{d\epsilon}\Big[\cancel{\bar{x}} + \epsilon\,\hat{x} + (\bar{y} + \epsilon\,\hat{y})^2\Big]_{\epsilon=0}$$

$$\hat{x} + 2(\bar{y} + \overset{2\;0}{\cancel{\epsilon\hat{y}}})\,\hat{y}\Big|_{\epsilon=0}$$

$$= \boxed{\hat{x} + 2\bar{y}\,\hat{y}}$$

Directional derivative of the residual

$$\underline{r} \rightarrow L_{\bar{\underline{h}}}\, \underline{r}(\hat{\underline{h}}) = 0 \rightarrow \hat{h} \text{ update}$$

$$= \underline{r}(\bar{\underline{h}}) + \epsilon \underbrace{\nabla_{\hat{\underline{h}}}\, \underline{r}(\bar{\underline{h}})}_{\text{find } = \underline{\underline{J}}(\underline{h})\, \hat{\underline{h}}}$$

$$r = \underline{\underline{D}}\left[\{\underline{\underline{M}}\,\underline{h}\}_f\, \underline{\underline{G}}\, \underline{h}\right] + f_s$$

$$\frac{d}{d\epsilon}\, \underline{r}(\bar{\underline{h}} + \epsilon \hat{\underline{h}})\Big|_{\epsilon=0} = \frac{d}{d\epsilon}\, \underline{\underline{D}}\left[\{\underline{\underline{M}}(\bar{\underline{h}} + \epsilon \hat{\underline{h}})\}_f\, \underline{\underline{G}}(\bar{\underline{h}} + \epsilon \hat{\underline{h}})\right] + \cancelto{0}{f_s}\Big|_{\epsilon=0}$$

$$= \frac{d}{d\epsilon}\, \underline{\underline{D}}\left[\left(\{\underline{\underline{M}}\,\bar{\underline{h}}\}_f + \epsilon \{\underline{\underline{M}}\,\hat{\underline{h}}\}_f\right)\left(\underline{\underline{G}}\,\bar{\underline{h}} + \epsilon\, \underline{\underline{G}}\,\hat{\underline{h}}\right)\right]\Big|_{\epsilon=0}$$

$$= \frac{d}{d\epsilon}\, \underline{\underline{D}}\Big[\cancelto{0}{\{\underline{\underline{M}}\,\bar{\underline{h}}\}_f\, \underline{\underline{G}}\,\bar{\underline{h}}} + \{\underline{\underline{M}}\,\bar{\underline{h}}\}_f\, \epsilon\, \underline{\underline{G}}\,\hat{\underline{h}} + \epsilon \{\underline{\underline{M}}\,\hat{\underline{h}}\}_f\, \underline{\underline{G}}\,\bar{\underline{h}}$$
$$+ \epsilon^2 \{\underline{\underline{M}}\,\hat{\underline{h}}\}_f\, \underline{\underline{G}}\,\hat{\underline{h}}\Big]\Big|_{\epsilon=0}$$

$$= \underline{\underline{D}}\Big[\{\underline{\underline{M}}\,\bar{\underline{h}}\}_f\, \underline{\underline{G}}\,\hat{\underline{h}} + \{\underline{\underline{M}}\,\hat{\underline{h}}\}_f\, \underline{\underline{G}}\,\bar{\underline{h}} + \cancelto{0}{2\epsilon}\, \underbrace{\{\underline{\underline{M}}\,\hat{\underline{h}}\}_f\, \underline{\underline{G}}\,\hat{\underline{h}}}_{\text{non-lin } \hat{\underline{h}}\, \underline{\underline{G}}\, \hat{\underline{h}}}\Big]\Big|_{\epsilon=0}$$

$$D_{\hat{\underline{h}}}\, \underline{r}(\bar{\underline{h}}) = \underline{\underline{D}}\left[\{\underline{\underline{M}}\,\bar{\underline{h}}\}_f\, \underline{\underline{G}}\,\hat{\underline{h}} + \{\underline{\underline{M}}\,\hat{\underline{h}}\}_f\, \underline{\underline{G}}\,\bar{\underline{h}}\right]$$

$$= \underline{\underline{J}}(\bar{\underline{h}})\, \hat{\underline{h}}$$

This is linear in $\underline{\hat{h}}$ but we need to pull $\underline{\hat{h}}$ out to form linear system.

$$\underbrace{\{\underline{\underline{M}}\,\underline{\hat{h}}\}}_{\underline{\hat{h}\text{mean}}}\;\underbrace{\underline{\underline{G}}\,\underline{\bar{h}}}_{\underline{d\bar{h}}} = \underline{\hat{h}\text{mean}} \;.*\; \underline{d\bar{h}} = \underline{d\bar{h}} \;.*\; \underline{\hat{h}\text{mean}} = \{$$

$$= \{\underline{\underline{G}}\,\underline{\bar{h}}\}_f\,\underline{\underline{M}}\,\underline{\hat{h}}$$

$$D_{\underline{\hat{h}}}\,\underline{r}(\underline{\bar{h}}) = \underbrace{D[\{\underline{\underline{M}}\,\underline{\bar{h}}\}_f\,\underline{\underline{G}} + \{\underline{\underline{G}}\,\underline{\bar{h}}\}_f\,\underline{\underline{M}}]}_{\underline{\underline{J}}(\underline{\bar{h}})}\;\underline{\hat{h}}$$

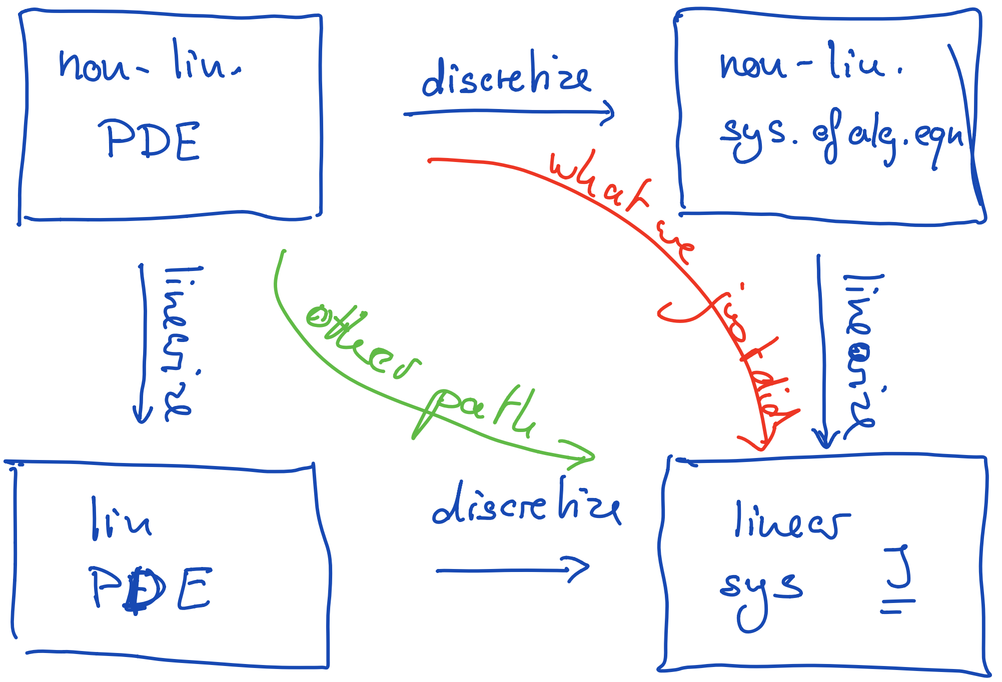
$$\boxed{\underline{\underline{J}} = \underline{\underline{D}}[\{\underline{\underline{M}}\,\underline{\bar{h}}\}_f\,\underline{\underline{G}} + \{\underline{\underline{G}}\,\underline{\bar{h}}\}_f\,\underline{\underline{M}}]}$$

Jacobian for steady unconfined flow.

Given $\underline{r}(\underline{h})$ and $\underline{\underline{J}}(\underline{h})$ and setting $\underline{\bar{h}} = \underline{h}^k$ and $\underline{\hat{h}} = \underline{dh}^k$ we have the Newton-Raphson update

$$\boxed{\begin{aligned} \underline{dh}^k &= -\underline{\underline{J}}(\underline{h}^k)^{-1}\,\underline{r}(\underline{h}^k) \\ \underline{h}^{k+1} &= \underline{h}^k + \underline{dh}^k \end{aligned}}$$

solve linear sys.



Linearize then discretize

$$r(h) = \nabla \cdot [h \nabla h] + 1$$

note $r(h)$ is a functional/operator

Need functional derivative / first variation

$$\boxed{D_{\hat{h}} r(\hat{h}) = \frac{d}{d\epsilon} r(\bar{h} + \epsilon \hat{h}) \Big|_{\epsilon=0} \approx \frac{r(\bar{h}+\epsilon\hat{h}) - r(\bar{h})}{\epsilon}}$$

(Gateaux derivative)

The linearization of functional

$$L_{\bar{h}} r(\hat{h}) = r(\bar{h}) + \epsilon D_{\hat{h}} r(\bar{h})$$

Note subscript & argument flipped

Linearize the eqn for unconfined flow

$$r = \nabla\cdot[h\nabla h] + 1$$

$$\nabla\bar{h} + \epsilon\nabla\hat{h}$$

$$\frac{d}{d\epsilon} r(\bar{h}+\epsilon\hat{h}) = \frac{d}{d\epsilon}\nabla\cdot\left[(\bar{h}+\epsilon\hat{h})\nabla(\bar{h}+\epsilon\hat{h})\right] + \cancelto{0}{1}\Big|_{\epsilon=0}$$

$$= \frac{d}{d\epsilon}\nabla\cdot\left[\cancelto{0}{\bar{h}\nabla\bar{h}} + \epsilon\bar{h}\nabla\hat{h} + \epsilon\hat{h}\nabla\bar{h} + \epsilon^2\hat{h}\nabla\hat{h}\right]\Big|_{\epsilon=0}$$

$$= \nabla\cdot\left[\bar{h}\nabla\hat{h} + \hat{h}\nabla\bar{h} + 2\epsilon\,\cancel{\hat{h}\nabla\hat{h}}\right]_{\epsilon=0}$$

$$D_{\hat{h}} r(\bar{h}) = \nabla\cdot\left[\bar{h}\nabla\hat{h} + \hat{h}\nabla\bar{h}\right]$$

$$= \underbrace{\nabla\cdot\left[\bar{h}\nabla + \nabla\bar{h}\right]}_{J(\bar{h}) \approx \underline{\underline{J}}(\underline{\bar{h}})}\ \bar{h}$$

Discretize linearized operator

$$J(\bar{h}) = \nabla\cdot\left[\bar{h}\nabla + \nabla\bar{h}\right]$$

$$\underline{\underline{J}}(\underline{\bar{h}}) = \underline{\underline{D}}\left[\{\underline{\underline{M}}\,\underline{\bar{h}}\}_f\,\underline{\underline{G}} + \{\underline{\underline{G}}\,\underline{\bar{h}}\}_f\,\underset{\substack{\uparrow\\ \text{not obvious}}}{\underline{\underline{M}}}\right]$$

$$\underbrace{N_f N_f N_f}_{N_f} N \qquad N_f N$$

$\Rightarrow$ get the same result