# Lecture 16: Unconfined flow continued

Logistics: - Please complete HW 5

- HW6 is up (Problems?)

erg_num
shift by $\frac{\Delta x}{2}$
Q2: add form of $\Pi=$

Last time: Numerical solution for steady unconfined flow

Continuous PDE: $-\nabla\cdot[h\,\nabla h] = 1$

Discrete residual: $\underline{r} = \underline{\underline{D}}\,[\{\underline{\underline{M}}\,\underline{h}\}_f\,\underline{\underline{G}}\,\underline{h}] - \underline{f_s}$

Directional derivative: $D_{\hat{\underline{h}}}\,\underline{r}(\underline{h}) = \frac{d}{d\epsilon}\,\underline{r}(\underline{h}+\epsilon\hat{\underline{h}})\Big|_{\epsilon=0} = \underline{\underline{J}}(\underline{h})\,\hat{\underline{h}}$

Jacobian for unconfined flow:

$$\boxed{\underline{\underline{J}} = \underline{\underline{D}}\,[\{\underline{\underline{M}}\,\bar{\underline{h}}\}_f\,\underline{\underline{G}} + \{\underline{\underline{G}}\,\bar{\underline{h}}\}_f\,\underline{\underline{M}}]}$$

Newton-Raphson method:

$$dh^k = -\,J(\underline{h}^k)^{-1}\,\underline{r}(\underline{h}^k)$$

$$h^{k+1} = h^k + dh^k$$

1 Discretize then linearize

2 Linearize then discretize

→ Functional derivative

get the same result

Today: Continue discussion of unconfined flow

- Flux computation
- Unconfined flow on spherical shell
- Numerical approximation of Jacobian
- Transient unconfined flow
- Example: Drainage of unconfined aquifer

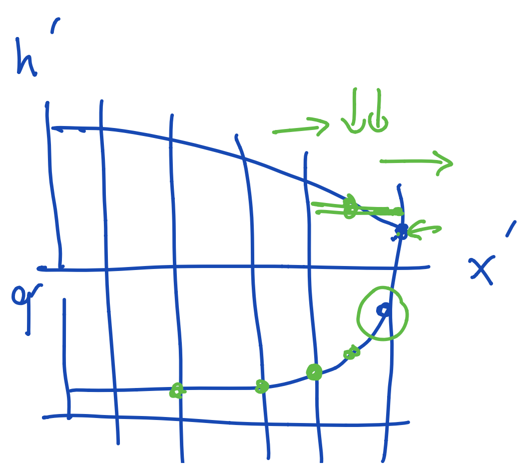
## Flux computation

$$-\nabla\cdot(h'\nabla h') = 1$$

$$\nabla\cdot(\underbrace{h'\underline{q}'}_{\underline{Q}}) = 1 \qquad \underline{q}' = -\nabla h'$$

$$\nabla\cdot\underline{Q} = 1$$

$$\frac{dQ}{dx} = 1$$


⇒ update compflux to compute $\underline{Q}' = h\underline{q}$

$\Rightarrow$ conservative out/inflow on bnd

(but to get q exact we need additional h unknown on bnd)

## Unconfined flow on spherical shell

$\Rightarrow$ because geometry is hidden in $\underline{\underline{D}}$, $\underline{\underline{G}}$ the Jacobian $\underline{\underline{J}}$ stays the same!

$\Rightarrow$ Newton-Raphson is no problem

Soln for steady unconfined flow with precip.

PDE: $$-\frac{1}{R\sin\theta}\frac{d}{d\theta}\left[\sin\theta\,\frac{k}{R}\,h\,\frac{dh}{d\theta}\right] = q_p \qquad \theta \in [0, \theta_b]$$

BC's: $$\left.\frac{dh}{d\theta}\right|_0 = 0 \qquad h(\theta_b) = h_b$$

$$h\frac{dh}{d\theta} = \frac{1}{2}\frac{dh^2}{d\theta} = \frac{1}{2}\frac{du}{d\theta}$$

$$u = h^2$$

dimensionless head: $h' = \frac{h}{h_c}$

$$-\frac{d}{d\theta}\left[\sin\theta\, h'\frac{dh'}{d\theta}\right] = \frac{q_p R^2}{k\, h_c^2} = 1 \Rightarrow \boxed{h_c = \sqrt{\frac{q_p R^2}{k}}}$$

Dimensionless problem:

PDE $\quad -\frac{d}{d\theta}\left[\sin\theta\, h' \frac{dh'}{d\theta}\right] = \sin\theta \qquad \theta \in [0, \theta_b]$

BC: $\quad \left.\frac{dh'}{d\theta}\right|_0 = 0 \qquad h'(\theta_b) = \Pi = \frac{h_b}{h_c}$

Integrate: $\quad -\sin\theta\, h' \frac{dh'}{d\theta} = -\cos\theta - c_1$

Neu. BC: $\quad 0 = -\cancel{\cos 0}^{\,1} - c_1 \quad \Rightarrow c_1 = -1$

$$-\sin\theta\, h' \frac{dh'}{d\theta} = -\cos\theta + 1$$

$$\sin\theta\, h' \frac{dh'}{d\theta} = \cos\theta - 1 \qquad q' = -\frac{dh'}{d\theta}$$

Integrate:

$$h'\,dh' = \left(\frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta}\right)d\theta = (\cot\theta - \csc\theta)\,d\theta$$

$$\frac{h'^2}{2} = \log(\cos\theta + 1) + c_2$$

Dir BC: $\quad \frac{\Pi^2}{2} = \log(\cos\theta_b + 1) + c_2$

$$c_2 = \frac{\Pi^2}{2} - \log(\cos\theta_b + 1)$$

$$\boxed{\begin{aligned} h' &= \sqrt{\Pi^2 + 2\log\left(\frac{\cos\theta + 1}{\cos\theta_b + 1}\right)} \\ q' &= -\frac{dh'}{d\theta} = \frac{1 - \cos\theta}{h'(\theta)\sin\theta} \end{aligned}}$$

$$Q' = h' q' = \frac{1-\cos\theta}{\sin\theta}$$

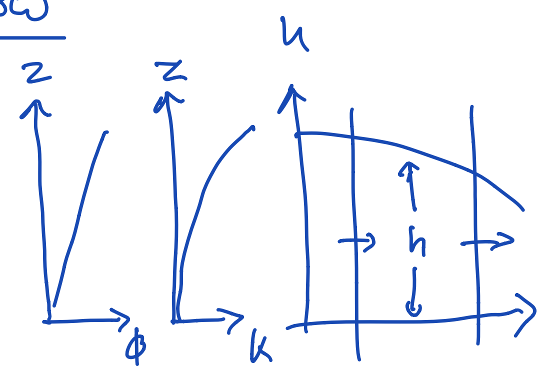
Sperical shell Newton see live script

Num. Newton see live script.

## Transient unconfined flow

(following Zhang et al 2013)

Lets consider an aquifer with vertical variation in $K(z) = K_0 z^n$ and $\phi(z) = \phi_0 z^m$




typically $n/m \in [2, 3]$

$\Rightarrow \quad K \sim \phi^2$ or $\phi^3$

The balance of fluid mass is

$$\frac{\partial}{\partial t} \int_0^h \phi \, dz + \nabla \cdot \int_0^h q(z) \, dz = f_s$$

where $h$ is water table $\quad q = -K(z) \nabla h$

Accumulation term

$$\int_0^h \phi(z)\,dz = \phi_o \int_0^h z^m\,dz = \frac{\phi_o}{m+1} z^{m+1}\Big|_0^h = \frac{\phi_o}{m+1} h^{m+1}$$

$$\frac{\partial}{\partial t}\int_0^h \phi(z)\,dz = \frac{\phi_o}{m+1}\underline{\frac{\partial h^{m+1}}{\partial t}} = \phi_o h^m \frac{\partial h}{\partial t} = \underline{\phi(h)}\frac{\partial h}{\partial t}$$

$\Rightarrow \phi(h) \rightarrow$ is unconfined storage

Flux term:

$$\nabla \cdot \int_0^h q(z)\,dz = -\frac{K_o}{n+1} h^{n+1} \nabla h$$

Governing equation

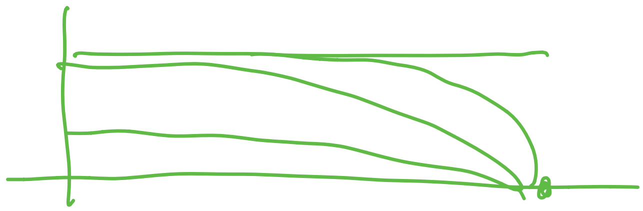
$$\boxed{\frac{\partial h^{m+1}}{\partial t} - D_h \nabla \cdot \left[h^{n+1} \nabla h\right] = f_s}$$

unconfined hydr. diff. : $D_h = \dfrac{K_o(m+1)}{\phi_o(n+1)}$

limiting case: $m=0$ $n=0$ $\Rightarrow \phi=\phi_o$ $K=K_o$

$$\frac{\partial h}{\partial t} \underbrace{- D_h \nabla \cdot [h \nabla h]}_{\text{steady term we have worked with}} = f_s$$

Next time



self-similar solution