Lecture 17: Transient unconfined flow continued

Logistics: - HW6 mostly done

- HW7 will be posted → Newton-Raphson

Last time: - Flux computation → $Q = hq$

- Unconfined flow on shell

→ essentially no change necessary

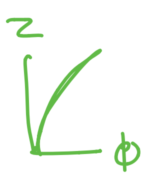
• analytic soln $h' = \sqrt{\pi^2 + 2\log\frac{\cos\theta + 1}{\cos\theta_0 + 1}}$

- Numerical Jacobian

→ good for testing

→ one-off problems

- Transient unconfined flow



$$\frac{\partial h^{m+1}}{\partial t} - D_h \nabla\cdot\left[h^{n+1}\nabla h\right] = f_s$$

$n = 3m$

$m \quad m$

$K \sim \phi^3$

Today: - Self-similar soln

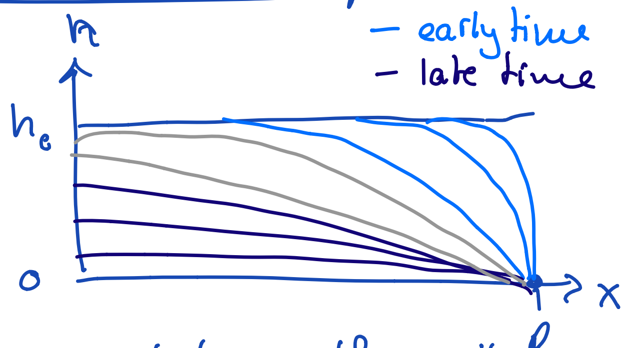
- Newton for self-similar ODE

- Jacobian for transient problem

Example: Drainage of an unconfined aquifer

PDE: $\frac{\partial h^{m+1}}{\partial t} + D_h \nabla \cdot [h^{n+1} \nabla h] = 0$

BC: $\nabla h \cdot \hat{n}|_0 = 0 \quad h(\ell) = 0$



This problem has an early and late self similar solution. Early solution is for semi-infinite aquifer → propagation of head front.

Late soln applies once head is dropping everywhere.

⇒ Late solution

Late self-similar solution (Zhang et al 2013)

External length scale: $x' = \frac{x}{\ell}$

External head scale: h_0 only relevant at early times

⇒ need to look for an internal head scale

really we are looking for a self-similar variable.

assume $h' = \frac{h}{h_c}$ $\quad t' = \frac{t}{t_c}$ substitute

$$\frac{h_c^{m+1}}{t_c}\frac{\partial h'^{m+1}}{\partial t'} - D_h \frac{h_c^{n+2}}{\ell^2}\nabla'\cdot\left[h'^{n+1}\nabla' h'\right] = 0$$

$$\frac{\partial h'^{m+1}}{\partial t'} - \underbrace{D_h \frac{t_c}{\ell^2} h_c^{n-m+1}}_{1}\nabla'\cdot\left[h'^{n+1}\nabla' h'\right] = 0$$

$$h_c = \left(\frac{\ell^2}{D_h t_c}\right)^{\frac{1}{n-m+1}} \qquad f = t^{\alpha} h$$

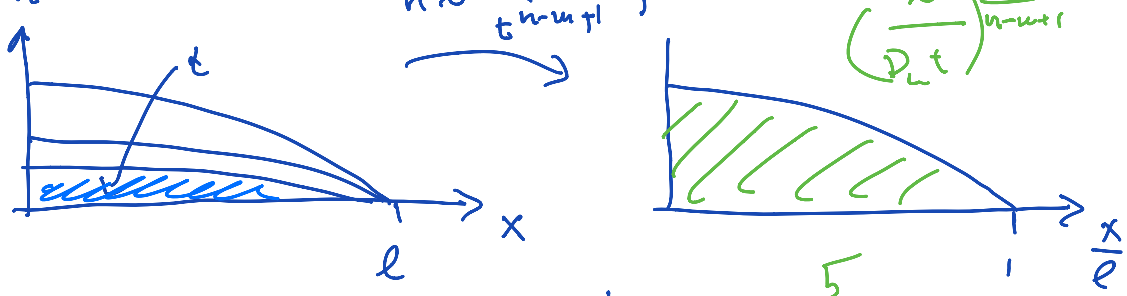
assume the self-similar variable has this form

$$\boxed{h = \left(\frac{\ell^2}{D_h t}\right)^{\frac{1}{n-m+1}} f\left(\frac{x}{\ell}\right)}$$

$f\left(\frac{x}{\ell}\right)$ is self-similar variable

$h \sim \frac{1}{t^{n-m+1}} f$

$\Pi_1 = \dfrac{h_0}{\left(\frac{\ell^2}{D_h t}\right)^{\frac{1}{n-m+1}}} \qquad \Pi_2 = \frac{x}{\ell}$



$m = n = 0 \Rightarrow \quad h \sim \frac{1}{t}$

Substitute this into PDE: $\rightarrow$ self-similar ODE:

$$\boxed{\begin{array}{ll} \text{ODE:} & \dfrac{d}{dx'}\left(f^{n+1}\dfrac{df}{dx}\right) + \dfrac{m+1}{n-m+1}\, f^{m+1} = 0 \\[2ex] \text{BC:} & \left.\dfrac{df}{dx'}\right|_0 = 0 \qquad f(1) = 0 \end{array}}$$

This non-linear ODE has no known analytic soln.

$\Rightarrow$ it must be solved numerically by Newton-Raphson

However, some important conclusions can be reached before even solving the ODE just from the form of the self. similar variable.

Consider the mass of GW in the aquifer.

$$M(t) = \rho w \int_0^{\ell} \underbrace{\int_0^{h} \phi \, dz}_{\frac{\phi_0}{m+1} h^{m+1}} dx = \frac{\rho w \phi_0}{m+1} \int_0^{\ell} h^{m+1} \, dz$$

substitute def. of similarity variable

$$h = \left(\frac{\ell^2}{D_w t}\right)^{\frac{1}{n-m+1}} \underbrace{f\left(\frac{x}{\ell}\right)}$$

$$s = \frac{x}{\ell} \qquad ds = \frac{dx}{\ell}$$

$$M(t) = \frac{\rho \omega \phi_0}{m+1} \int_0^1 \left[\left(\frac{\ell^2}{D_w t}\right)^{\frac{1}{n-m+1}} f(s)\right]^{m+1} \ell \, ds$$

$$\boxed{M(t) = \frac{\rho \omega \phi_0 \ell}{m+1} \left(\frac{\ell^2}{D_w t}\right)^{\frac{m+1}{n-m+1}} \underbrace{\int_0^1 f(s)^{m+1} ds}_{\text{just a number} \sim 1}}$$

$$M \sim \frac{1}{t^{\frac{m+1}{n-m+1}}}$$

Exampe $m = n = 0$: $\quad M \sim \frac{1}{t}$

$m = 1 \quad n = 4$: $\quad M \sim \frac{1}{t^{\frac{2}{3}}}$

$\Longrightarrow$ get the exponent of decline