Lecture 18: Numerical solution unconfined flow

Logistics: - HW 6 ✓

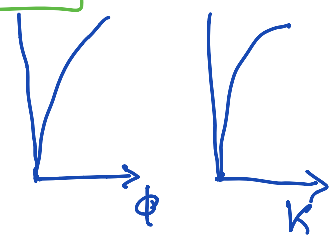
- no new HW this week (sorry)

Last time: Transient unconfined flow

$$\boxed{\frac{\partial h^{m+1}}{\partial t} - D_h \nabla \cdot \left[h^{n+1} \nabla h \right] = 0}$$

$\phi = \phi_0 z^m \qquad K = K_0 z^m$

$$\boxed{D_h = \frac{K_0 (m+1)}{\phi_0 (n+1)}}, \quad \frac{n}{m} \in [2,3]$$



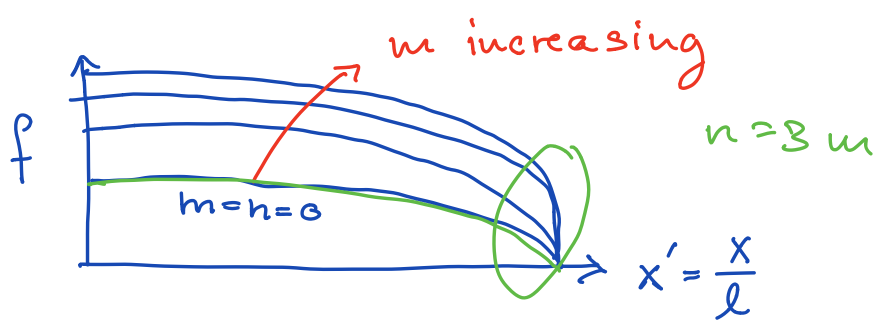
- Late self-similar soln for unconfined drainage $\Rightarrow$

$$\boxed{f\left(\frac{x}{\ell}\right) = \frac{h(x,t)}{\left(\frac{\ell^2}{D_h t}\right)^{\frac{1}{n-m+1}}}}$$

Zheng 2013

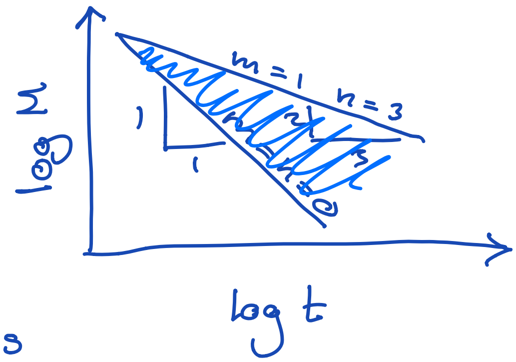
- ODE: $\frac{d}{dx'}\left(f^{n+1} \frac{df}{dx'} \right) + \frac{m+1}{n-m+1} f^{m+1} = 0$

$\left.\frac{df}{dx'}\right|_0 = 0 \qquad f(1) = 0$



- Mass decline in aquifer

$$\boxed{M(t) \sim \frac{1}{t^{\frac{m+1}{n-m+1}}}}$$



⇒ Decline in ϕ & K with depth slows drainage of the aquifer

Today: - Numerical solution transient problem

- Jacobian
- Integration of Newton with time stepping

- Comparison of transient and self-similar solutions
- Apply results to Martian highlands aquifer
  - What to choose for m & n?
  - Spherical shell geometry
  - $\boxed{\text{Time scales?}}$

Numerical solution transient unconfined flow

PDE: $\underbrace{\phi_0 h^m}_{\phi(h)} \frac{\partial h}{\partial t} - \nabla \cdot \left[\frac{k_0}{n+1} h^{n+1} \nabla h \right] = 0$

lets write this as generic non-linear diff. problem

$$\boxed{s(u) \frac{\partial u}{\partial t} - \nabla \cdot [f(u) \nabla u] = f_s}$$

here $s(u)$ and $f(u)$ are arbitrary differentiable functions: ($s = \phi_0 u^m$ $\quad f = \frac{k_0}{n+1} u^{n+1}$)

$s \sim$ storativity / heat capacity

f – conductivity

Discretization with Backward Euler

$$\{\underline{s}(\underline{u}^{n+1})\}_c (\underline{u}^{n+1} - \underline{u}^n) - \Delta t \, \underline{\underline{D}} \left[\{\underline{\underline{M}} \, \underline{f}(\underline{u}^{n+1})\}_f \, \underline{\underline{G}} \, \underline{u}^{n+1} \right] = \Delta t \, \underline{f_s}$$

Discrete residual: $\underline{u}^{n+1} = \underline{u}$

$$\underline{r}(\underline{u}, \underline{u}^n) = \{\underline{s}(\underline{u})\}_c (\underline{u} - \underline{u}^n) - \Delta t \underline{\underline{D}} \left[\{\underline{\underline{M}} \, \underline{f}(\underline{u})\}_f \, \underline{\underline{G}} \, \underline{u} \right] - \Delta t \, \underline{f_s}$$

Find $\underline{u}$ s.t. $\quad \underline{r}(\underline{u}, \underline{u}^n) = 0$

Linearize $\underline{\Gamma}$ $\quad \underline{u} = \underline{\bar{u}} + \epsilon \underline{\hat{u}} \qquad k=0 \quad \underline{\bar{u}} = \underline{u}^n$

Directional derivative: $D_{\hat{u}}\, \underline{\Gamma}(\underline{\bar{u}})$

$$\frac{d}{d\epsilon} \underline{\Gamma}(\underline{\bar{u}} + \epsilon \underline{\hat{u}})\Big|_{\epsilon=0} = \frac{d}{d\epsilon} \underbrace{\{\underline{s}(\underline{\bar{u}} + \epsilon \underline{\hat{u}})\}_c (\underline{\bar{u}} + \epsilon \underline{\hat{u}} - \underline{u}^n)}_{acc} \underbrace{- \Delta t\, \underline{\underline{D}} \Big[\{\underline{\underline{M}}\, \underline{f}(\underline{\bar{u}} + \epsilon \underline{\hat{u}})\}_f \; \underline{\underline{G}}(\underline{\bar{u}} + \epsilon \underline{\hat{u}}) \Big]_0 \cancel{- \Delta t f_s}}_{flux}$$

acc:

$$\left\{ \frac{ds}{d\underline{u}} (\underline{\bar{u}} + \cancel{\epsilon \underline{\hat{u}}})\, \hat{u} \right\}_c (\underline{\bar{u}} + \cancel{\epsilon \underline{\hat{u}}} - \underline{u}^n) + \{\underline{s}(\underline{\bar{u}} + \cancel{\epsilon \underline{\hat{u}}})\}_c\, \underline{\hat{u}} \;\Big|_{\epsilon=0}$$

$$\left\{ \frac{ds}{d\underline{u}} (\bar{u})\, \hat{u} \right\}_c (\underline{\bar{u}} - \underline{u}^n) + \{\underline{s}(\underline{\bar{u}})\}_c\, \underline{\hat{u}} \qquad \text{linear in } \hat{u}$$

$$\underbrace{\left\{ \frac{ds}{d\underline{u}} (\underline{\bar{u}}) \right\}_c}_{\underline{\underline{dS}}(\underline{\bar{u}})} \underbrace{\{\underline{\bar{u}} - \underline{u}^n\}_c}_{\underline{U}(\underline{\bar{u}}, \underline{u}^n)} \underline{\hat{u}} + \underbrace{\{\underline{s}(\underline{\bar{u}})\}_c}_{\underline{\underline{S}}(\underline{\bar{u}})} \underline{\hat{u}}$$

$$acc = \underbrace{\left[\underline{\underline{dS}}(\underline{\bar{u}})\, \underline{U}(\underline{\bar{u}}, \underline{u}^n) + \underline{\underline{S}}(\underline{\bar{u}}) \right]}_{\underline{\underline{J}}_{acc}} \underline{\hat{u}}$$

$$\mathrm{flux} := \Delta t \, \underline{\underline{D}} \left[\left\{ \underline{\underline{M}} \, \underline{f}(\underline{\bar{u}} + \epsilon \hat{\underline{u}}) \right\}_f \underline{\underline{G}} \, (\underline{\bar{u}} + \epsilon \hat{\underline{u}}) \right]$$

$$\Delta t \, \underline{\underline{D}} \left[\left\{ \underline{\underline{M}} \, \frac{df}{du}(\underline{\bar{u}} + \cancelto{0}{\epsilon \hat{\underline{u}}}) \, \hat{\underline{u}} \right\}_f \underline{\underline{G}} \, (\underline{\bar{u}} + \cancelto{0}{\epsilon \hat{\underline{u}}}) + \left\{ \underline{\underline{M}} \, f(\underline{\bar{u}} + \cancelto{0}{\epsilon \hat{\underline{u}}}) \right\}_f \underline{\underline{G}} \, \hat{\underline{u}} \right] \Bigg|_{\varepsilon = 0}$$

$$\Delta t \, \underline{\underline{D}} \left[\underbrace{\left\{ \underline{\underline{G}} \, \underline{\bar{u}} \right\}_f}_{\underline{\underline{GU}}(\underline{\bar{u}})} \underbrace{\left\{ \underline{\underline{M}} \, \frac{df}{du}(\underline{\bar{u}}) \, \hat{\underline{u}} \right\}_f}_{?} + \underbrace{\left\{ \underline{\underline{M}} \, f(\underline{\bar{u}}) \right\}_f}_{\underline{\underline{F}}(\underline{\bar{u}})} \underline{\underline{G}} \, \hat{u} \right]$$

$$\left\{ \underline{\underline{M}} \, \frac{df}{du}(\underline{\bar{u}}) \, \hat{u} \right\}_f = \underset{N_f \, N}{\underline{\underline{M}}} \left\{ \overset{df \, .* \, \hat{\underline{u}}}{\underset{N \cdot N}{\frac{df}{du}(\underline{\bar{u}}) \, \hat{\underline{u}}}} \right\}_c = \underline{\underline{M}} \underbrace{\left\{ \frac{df}{du}(\underline{\bar{u}}) \right\}_c}_{\underline{\underline{dF}}(\underline{\bar{u}})} \hat{\underline{u}}$$

$$D_{\hat{u}} \, \mathrm{flux}(\underline{\bar{u}})$$

$$\cancel{J_{flux}} = \underbrace{\Delta t \, \underline{\underline{D}} \left[\underline{\underline{GU}}(\underline{\bar{u}}) \, \underline{\underline{M}} \, \underline{\underline{dF}}(\underline{\bar{u}}) + \underline{\underline{F}}(\underline{\bar{u}}) \, \underline{\underline{G}} \right]}_{J_{flux}} \hat{\underline{u}}$$

Jacobian for non-linear diffusion:

$$\boxed{J(\underline{u}, \underline{u}^n) = \underline{\underline{dS}}(\bar{u}) \, \underline{U}(\underline{u}, \underline{u}^n) + \underline{\underline{S}}(\underline{u}) - \Delta t \, \underline{\underline{D}} \left[\underline{\underline{GU}}(\underline{u}) \, \underline{\underline{M}} \, \underline{\underline{dF}}(\underline{\bar{u}}) + \underline{\underline{F}}(\underline{\bar{u}}) \, \underline{\underline{G}} \right]}$$

Implementation outline.

```
for   n = 1 : Nt              % time stepping loop

      uold = u ;          % uold = u^n

      while   nres > tol || ndu > tol || k < kmax

            du = - J(u, uold)^-1 res(u, uold);

            u = u + du ;
            k = k+1;
      end

end
```