Lecture 23: Ocean - Groundwater Interaction

Logistics: - HW8 all done ✓

- Fill out course evaluation → $\int$ix

Last time: 2D sperical shell discretization

$$\nabla h = \frac{1}{R}\frac{\partial h}{\partial \theta}\hat{\theta} + \frac{1}{R\sin\theta}\frac{\partial h}{\partial \varphi}\hat{\varphi}$$

$$\nabla\cdot q = \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, q_\theta) + \frac{1}{R\sin\theta}\frac{\partial q_\varphi}{\partial \varphi}$$

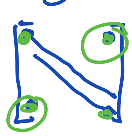
x-dir: modify 1D op. then kron

1D

1D cap → $\boxed{Dx = R\_sin\_c\_inv * D * Sin\_f}$, $Gx = Gx / R$

$Dx = \text{kron}(Dx, Iy)$, $Gx = \text{kron}(Gx, Iy)$

y-dir: periodicity & modification during kron

1D $Gy =$  periodic BC in 1D

$Dy = \text{kron}(R\_sin\_c\_inv, Dy)$

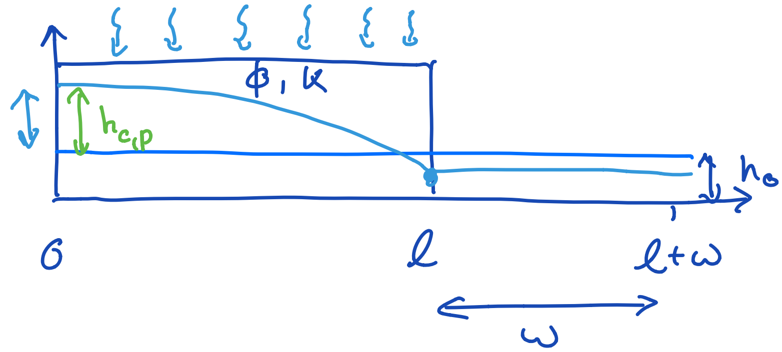
$Dx = \text{kron}(R\_sin\_c\_inv, Gy)$

Today: Start thinking about interaction between GW and ocean

Ocean - Groundwater interaction

On global scale the total volume of water is fixed.

$$V = V_G + V_O = \text{const.}$$



Consider the following steady, linear, unconfined problem.

PDE: $-\nabla \cdot [k h \nabla h] = q_p$ on $x \in [0, \ell]$

BC: $q \cdot \hat{n}|_0 = 0$ $\quad h(\ell) = h_o = \frac{V}{\omega} - \frac{\phi}{\omega}\int_0^\ell h(x)\,dx$

C: $\boxed{V = V_G + V_O}$ $\quad V_G = \int_0^\ell \phi\, h(x)\,dx$ $\quad V_O = h_o\,\omega$

Determine h_o: $\quad V = h_o\,\omega + \phi\int_0^\ell h\,dx$

$$h_o = \frac{V}{\omega} - \frac{\phi}{\omega}\int_0^\ell h\,dx$$

Dimensional problem

PDE: $\quad -\nabla \cdot [k h \nabla h] = q_p$ on $x \in [0.$

Introduce $h' = \frac{h}{h_c}$ $\quad x' = \frac{x}{\ell}$ $\quad q' = \frac{q}{q_c}$ $\quad q_c = k\frac{h_c}{\ell}$

Substitute into PDE:

$$-\frac{K h_c^2}{\ell^2} \nabla' \cdot [h' \nabla' h'] = q_p$$

$$-\nabla' \cdot [h' \nabla' h'] = \frac{q_p \ell^2}{K h_c} \rightarrow h_c$$

increase in h due to precip

Subst. into BC:

$dx = d(\ell x') = \ell dx'$

$$h_c h'(1) = \frac{V}{\omega} - \frac{\phi}{\omega} h_c \ell \int_0^1 h' dx$$

$$h'(1) = \frac{V}{\omega h_c} - \frac{\phi \ell}{\omega} \int_0^1 h' dx$$

$\rightarrow h_c$ sea level if all water is in ocean

Third h scale: water height in absence of precip.

$$h(x) = h_o : \quad V = \phi h_o \ell + \omega h_o = (\phi \ell + \omega) h_o$$

$$\boxed{h_o = \frac{V}{\phi \ell + \omega} \equiv h_c}$$

Substitute this into PDE:

$$-\nabla' \cdot [h' \nabla' h'] = Pr \qquad Pr = \frac{h_p}{h_o} = \frac{q_p \ell^2 (\phi \ell + \omega)}{K V}$$

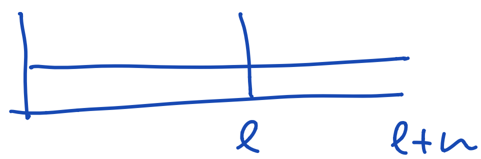
Interpretation: change in h due to precip

relative to water level without precip

Substitute h_c into BC:

$$h'(1) = \frac{V}{\omega h_c} - \frac{\phi \ell}{\omega}\int_0^1 h' \, dx'$$
$$= \frac{\cancel{V}(\phi\ell + \omega)}{\omega \cancel{V}} - \frac{\phi\ell}{\omega}\int_0^1 h' \, dx'$$
$$= 1 + \frac{\phi\ell}{\omega} - \frac{\phi\ell}{\omega}\int_0^1 h' \, dx' = 1 + \frac{\phi\ell}{\omega}\left(1 - \int_0^1 h' \, dx'\right)$$

$Ca = \dfrac{\omega}{\phi\ell}$ "Capacity number of ocean"



$Ca = \dfrac{V_0}{V_{GW}}$ for $h = \text{const}$

Dimensionless problem (dropping primes)

$$\boxed{\begin{array}{ll} \text{PDE:} & -\nabla\cdot[h\nabla h] = Pr \qquad x \in [0, 1] \\ \text{BC:} & q\cdot \vec{n}\big|_0 = 0 \qquad h(1) = \Pi_0 = 1 + \frac{1}{Ca}\left(1 - \int_0^1 h\,dx\right) \end{array}}$$

Idea: First solve for shape of GW, $h(x, \Pi_0)$, assuming Π_0 is a free parameter. Then, determine Π_0 from mass balance.

Integrate: $-h\frac{du}{dx} = Pr\, x + c_1$

Neu. BC: $0 = Pr\, 0 + c_1 \Rightarrow c_1 = 0$

Integrate: $-h\,dx = Pr\, x\, dx$

$$-\frac{h^2}{2} = Pr\,\frac{x^2}{2} + c_2$$

Dir. BC: $-\frac{\Pi_0^2}{2} = Pr\,\frac{1}{2} + c_2 \Rightarrow c_2 = -\frac{1}{2}(Pr + \Pi_0^2)$

$$\Rightarrow \boxed{h = \sqrt{\Pi_0^2 + Pr(1-x^2)}}$$

Now we need to determine Π_0 from overall mass conservation.

$$h(1) = \boxed{\Pi_0 = 1 + \frac{1}{Ca}\left(1 - \int_0^1 h(x, \Pi_0, Pr)\,dx\right)} = obj(Ca)$$

Find Π_0 so that equ is satisfied

Hence we need $H(\Pi_0, Pr) = \int_0^1 h\,dx$

$$H(\Pi_0, Pr) = \frac{\sqrt{Pr}\,(Pr + \Pi_0^2)\,\mathrm{asin}\sqrt{\frac{Pr}{Pr+\Pi_0^2}} + \Pi_0 Pr}{2Pr}$$

$\frac{\Pi_0}{2}$

We know $\lim_{Pr \to 0} H = \underline{\Pi_0}$

$\operatorname{asin}(x) \approx x \quad |x| \ll 1$

$\Rightarrow$ Matlab