Lecture 5: Boundary conditions

Logistics: - HW1   P1   6/9   P2   5/9

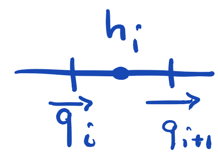
due on Thursday

feel free to work together! Use Piazza!

- HW2 is posted due next Thursday

Last time: - Conservative finite differences

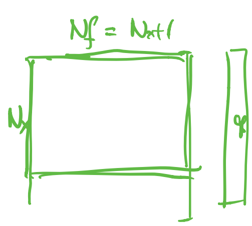
- staggered grid



- Discrete operators

$-\nabla\cdot[K\nabla h] = f_s$ continuum

$\nabla\cdot q = f_s$ $\boxed{-\underline{\underline{D}} * [K\,\underline{\underline{G}}\,\underline{h}] = \underline{f_s}}$ discrete



$\underline{\underline{D}}$ = discrete divergence matrix $N_x{+}1 \cdot N_x$

$\underline{\underline{G}}$ = discrete gradient matrix $N_x \cdot N_x{+}1$

$\Rightarrow$ sparse diagonals

- adjoint relation: $\boxed{\underline{\underline{G}} = -\underline{\underline{D}}^T}$ (in interior)

Today: - Forward testing of operators

- Dirichlet BC $\leftrightarrow$ Constraints

Dirichlet BC's and constraints

→ BC is required for problem to be well posed

$$\underline{\underline{L}} = -\underline{\underline{D}} * K * \underline{\underline{G}} \qquad \underline{\underline{L}}\,\underline{h} = \underline{f}_s$$

$$\underline{h} = \underline{\underline{L}}^{-1} \underline{f}_s$$

solve lin. sys. with backslash: $\boxed{\underline{h} = \underline{\underline{L}} \backslash \underline{f}_s}$

not scalable ! $\cancel{\underline{h} = inv(\underline{\underline{L}}) * \underline{f}_s}$

is not invertible is singular

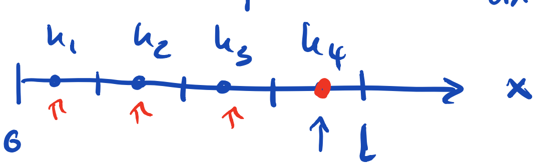
⇒ we need BC' to make problem well posed.

Example: Highland aquifer

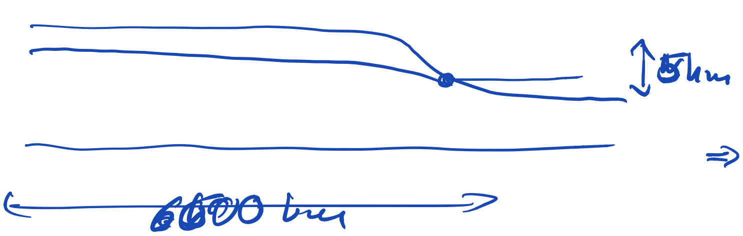
PDE: $\quad -\frac{d}{dx}\left(bK \frac{dh}{dx}\right) = f_s \qquad x \in [0, L]$

BC: $\quad h(L) = h_0 \equiv 0 \qquad$ (homog. Dirichlet BC)

$q(0) = 0 \Rightarrow \left.\frac{dh}{dx}\right|_0 = 0 \qquad$ (homog. Neumann BC ⇒ natural BC)



note: Dir. BC is imposed

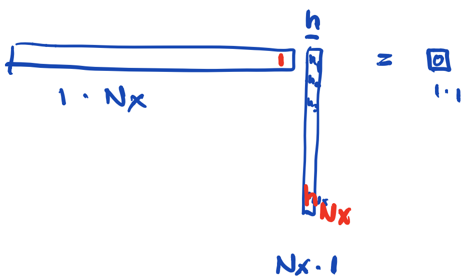


at cell center → small error

⇒ Dir BC's are constraints

Need to write BC as a lin. system

$\underline{\underline{B}}\ \underline{h} = \underline{0}$ ~~let~~ $\boxed{h_{Nx} = 0}$



$\cancel{h_1 - h_2 = 0}$

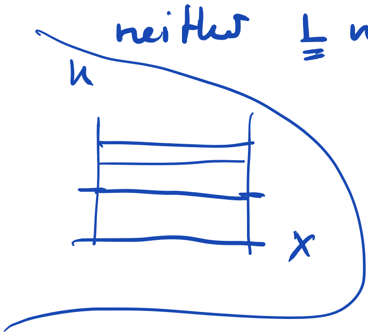
B is a N_e by N_x matrix, where Ne is number of constraints (= cells with Dir. BC)

Full statement of discrete problem:

PDE : $\underline{\underline{L}}\ \underline{h} = \underline{f_s}$ $\underline{\underline{L}} = -\underline{\underline{D}} * k * \underline{\underline{G}}$ system matrix $Nx \cdot Nx$

BC : $\underline{\underline{B}}\ \underline{h} = 0$ $\underline{\underline{B}}$ is constraint matrix $Ne \cdot Nx$

neither $\underline{\underline{L}}$ nor $\underline{\underline{B}}$ are invertible, i.e. unique solutions.



$\Rightarrow$ need to combine $\underline{\underline{L}}$ and $\underline{\underline{B}}$ to form <u>reduced</u> linear system

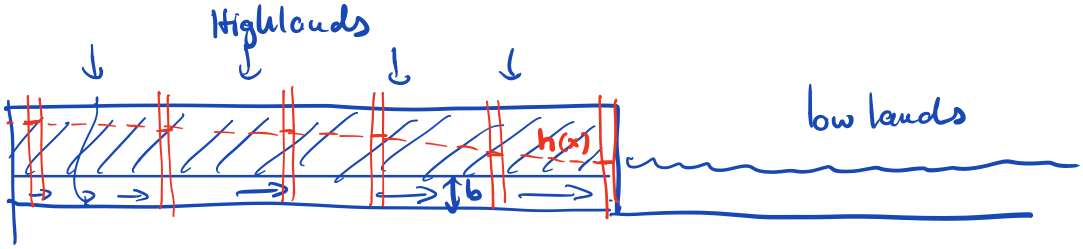
$$\boxed{\underline{\underline{L}}_r\ \underline{h}_r = \underline{f}_{s,r}}$$

How do we form these from info above

$\underline{\underline{L}}_r$ is $(Nx - Ne) \cdot (Nx - Ne)$

$\underline{h}_r$ is $(Nx - Ne) \cdot 1$

$\underline{f}_{sr}$ is $(Nx - Ne) \cdot 1$



1 starter problem → implement BC

2) → cylindrical symmetry

3) → spherical shell

⇒ update operators

still linear

4) unconfined aquifer. → non-linear