Lecture 6: Scaling Analysis & Dirichlet BC's

Logistics: - HW 1 is due today ✓

- HW 2 is posted due next Thursday

Last time: - Example problem: Southern highlands aquifer

- Writing the Dir BCs as a linear system

$\underline{\underline{B}}\,\underline{h} = \underline{0}$ $N_e \cdot N_x$

- Full discrete problem: $\boxed{\begin{array}{l} \underline{\underline{L}}\,\underline{h} = \underline{f}_s \\ \underline{\underline{B}}\,\underline{h} = \underline{0} \end{array}}$

Example problem: Linear confined aquifer

PDE $\quad -\dfrac{d}{dx}\left(bk\dfrac{dh}{dx}\right) = f_s \qquad x \in [0, \ell]$

BC: $\quad \left.\dfrac{dh}{dx}\right|_0 = 0 \qquad h(\ell) = h_0$

depend. variable: h $\qquad$ independ. variable: x

Parameters: $\underline{b, \overset{\phi}{K}, f_s, \underline{\ell}, h_0}$ (5)

To determine # of indep. parameters we non-dimensionalize the variables with the parameters.

Dimensionless variables:

$$x' = \frac{x}{\underset{\substack{\uparrow \\ \text{external scale}}}{\ell}} \qquad x' \in [0,1] \qquad\qquad h' = \frac{h - h_0}{\underset{\substack{\uparrow \\ \text{internal scale to be determined}}}{h_c}}$$

substitute into PDE & BC : $x = \ell x'$ $\quad h = h_0 + h_c h'$

PDE: $$-\frac{d}{d(\ell x')}\left[bK \frac{d(h_0 + h_c h')}{d(\ell x')}\right] = f_s \qquad \cancel{\ell} x' \in [0, \cancel{\ell}^{1})$$

$$-\frac{1}{\ell}\frac{d}{dx'}\left[\frac{bK}{\ell}\left(\cancel{\frac{dh_0}{dx'}}^{0} + h_c \frac{dh'}{dx'}\right)\right] = f_s$$

$$-\frac{bKh_c}{\ell^2}\frac{d^2h'}{dx'^2} = f_s$$

$$-\underbrace{\frac{d^2h'}{dx'^2}}_{\text{dim. less}} = \underbrace{\frac{f_s \ell^2}{bK h_c}}_{\text{dim. less}} = 1$$

l.h.s. suggest an internal head scale: $h_c = \dfrac{f_s \ell^2}{bK}$

$\Rightarrow$ dim less PDE $\qquad -\dfrac{d^2h'}{dx'^2} = 1 \qquad x' \in [0,1]$

BC: $$\left.\frac{dh}{dx}\right|_{x=0} = \frac{\cancel{h_c}}{\cancel{\ell}}\left.\frac{dh'}{dx'}\right|_{\ell x'=0} = 0 \quad \Rightarrow \quad \left.\frac{dh'}{dx'}\right|_0 = 0$$

$$h(x=\ell) = h_0$$

$$\cancel{h_0} + \cancel{h_c}\, h'(x'\cancel{\ell} = \cancel{\ell}^{1}) = \cancel{h_0}\ 0 \qquad \Rightarrow \quad h'(1) = 0$$

Dimensionless problem: PDE $\quad \boxed{\begin{array}{ll} -\dfrac{d^2h'}{dx'^2} = 1 & x' \in [0,1] \\ \dfrac{dh'}{dx'}\Big|_0 = 0 & h'(1) = 0 \end{array}}$

BC:

no parameter left

Analytic solution:

integrate once: $\quad -\dfrac{dh'}{dx'} = x' + c_1$

use 1^{st} BC: $\quad -\dfrac{dh'}{dx'}\Big|_0 = 0 + c_1 = 0 \quad \Rightarrow \quad c_1 = 0$

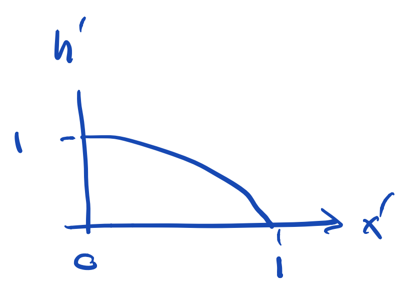
$$-\frac{dh'}{dx'} = x'$$

integrate second time: $\quad -h' = \dfrac{x'^2}{2} + c_2$

use 2^{nd} BC: $\quad -h(1) = \dfrac{1}{2} + c_2 = 0 \quad \Rightarrow \quad c_2 = -\dfrac{1}{2}$

$$\Rightarrow \quad -h' = \frac{x'^2}{2} - \frac{1}{2}$$

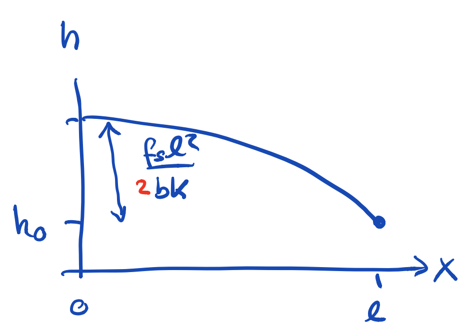
dimensionless soln: $\quad \boxed{h' = \frac{1}{2}(1 - x'^2)}$



substitute to redimensionalize

$$x' = \frac{x}{\ell} \qquad h' = \frac{h - h_0}{h_c} \qquad h_c = \frac{f_s \ell^2}{bk}$$

$$\boxed{h = h_0 + \frac{f_s \ell^2}{2bk}\left(1 - \left(\frac{x}{\ell}\right)^2\right)}$$



Hence the internal head scale $\frac{f_s \ell^2}{bK}$ gives the order of magnitude of the increase in head across aquifer.

Dirichlet BC (Homogeneous)

The discrete dimensionless problem

1) PDE $\quad \underline{\underline{L}}\, \underline{h} = \underline{f_s} \qquad \underline{\underline{L}} = -\underline{\underline{D}} * \underline{\underline{G}} \qquad N_x \cdot N_x$

2) BC $\quad \underline{\underline{B}}\, \underline{h} = \underline{0} \qquad \underline{\underline{B}} = [\cdots \quad 1] \quad N_c \cdot N_x$

Need to combine them into a reduced system. with $N_x - N_c$.

Reduced linear system: $\boxed{\underline{\underline{L}}_r \underline{h}_r = \underline{f}_{sr}}$

$\underline{h}_r \quad N_x - N_c \cdot 1$

$\underline{f}_{sr} \quad N_x - N_c \cdot 1$

$\underline{\underline{L}}_r \quad (N_x - N_c) \cdot (N_x - N_c)$

Projection matrix

What is the relation between: $\underline{h}_r$ and $\underline{h}$?

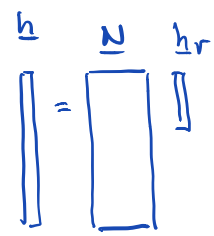
$\underline{f}_{sr}$ and $\underline{f}_s$?

$\underline{\underline{L}}_r$ and $\underline{\underline{L}}$?

Remember everything is linear.

$\Rightarrow$ two vectors of different length are related by a rectangular matrix.

$$\underset{N_x \cdot 1}{\underline{h}} = \underset{N_x \cdot (N_x - N_c)}{\underline{\underline{N}}} \; \underset{(N_x - N_c) \cdot 1}{\underline{h}_r}$$



What is $\underline{\underline{N}}$?

For now we just require that $\underline{\underline{N}}$ is orthonormal.

If $\underline{n}_i$ is the i-th column of $\underline{\underline{N}} = [\underline{n}_1 \; \underline{n}_2 \; \underline{n}_3]$

then $\underline{n}_i^T \cdot \underline{n}_i = 1$ for all i (normal)

$\underline{n}_j^T \cdot \underline{n}_i = 0$ $j \neq i$ (ortho)

Then it follows:

a) $\underset{(N_x - N_c) \cdot N_x}{\underline{\underline{N}}^T} \; \underset{N_x \cdot (N_x - N_c)}{\underline{\underline{N}}} = \underset{(N_x - N_c) \cdot (N_x - N_c)}{\underline{\underline{I}}_r}$ identity in reduced space

b) $\underset{N_x \cdot (N_x - N_c)}{\underline{\underline{N}}} \quad \underset{(N_x - N_c) \cdot N_x}{\underline{\underline{N}}^T} = \underset{N_x \cdot N_x}{\underline{\underline{I}}'}$ "identity" matrix in full space but N_c zeros on diagonal

If this is the case: $\underline{h} = \underline{\underline{N}}\, \underline{h}_r$

$$\underline{\underline{N}}^T \underline{h} = \underbrace{\cancel{\underline{\underline{N}}^T \underline{\underline{N}}}}_{\underline{\underline{I}}_r} \underline{h}_r = \underline{h}_r$$

$$\boxed{\begin{aligned} \underline{h} &= \underline{\underline{N}}\, \underline{h}_r \\ \underline{h}_r &= \underline{\underline{N}}^T \underline{h} \end{aligned}}$$

$$\underline{h}_r = \underline{\underline{N}}^T \underline{h}$$

here $\underline{\underline{N}}$ is a matrix that allows us to go between full & reduced space. We say $\underline{\underline{N}}^T$ projects vector of unknowns into the reduced space.

Similarly: $\underline{f}_s = \underline{\underline{N}}\, \underline{f}_{sr} \qquad \underline{f}_{sr} = \underline{\underline{N}}^T \underline{f}_s$

How is the system matrix projected into the reduced space?

$$\underline{\underline{L}}\, \underline{h} = \underline{f}_s$$

$$\underline{\underline{N}}^T \underline{\underline{L}}\, \underline{h} = \underline{\underline{N}}^T \underline{f}_s = \underline{f}_{sr}$$

insert $\underline{\underline{I}}' = \underline{\underline{N}}\, \underline{\underline{N}}^T$ on left

$$\underbrace{\underline{\underline{N}}^T \underline{\underline{L}}\, \underline{\underline{N}}}_{\underline{\underline{L}}_r} \underbrace{\underline{\underline{N}}^T \underline{h}}_{\underline{h}_r} = \underline{f}_{sr}$$

$$\underline{\underline{L}}_r \, \underline{h}_r = \underline{f}_{sr}$$

$$\Rightarrow \quad \boxed{\underbrace{\underline{\underline{L}}_r}_{} = \underbrace{\underline{\underline{N}}^T}_{N_x - N_c \cdot N_x} \underbrace{\underline{\underline{L}}}_{N_x \cdot N_x} \underbrace{\underline{\underline{N}}}_{N_x \cdot N_x - N_c}}$$

$\underline{\underline{L}}_r$ is $N_x - N_c \cdot N_x - N_c$

Now we just need to find $\underline{\underline{N}}$!

Null space of the constraint matrix

In which space should we look for the solution?

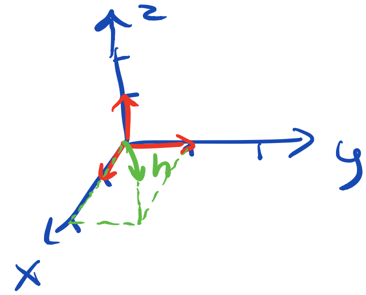
$\Rightarrow$ Any solution that satisfies the BC, ie constraints

All $\underline{h}$ that satisfy $\underline{\underline{B}}\,\underline{h} = \underline{0}$ all vectors that are zero on right bnd.

$\Rightarrow$ this is the null space $\mathcal{N}(\underline{\underline{B}})$ of the constraint matrix.

The matrix $\underline{\underline{N}}$ can be <u>any</u> orthonormal basis for $\mathcal{N}(\underline{\underline{B}})$. There are many possible bases.

A basis is a set of vectors that allows you to access any point within the vector space via linear combination.



$$\underline{N} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \qquad h = \begin{pmatrix}2\\1\\1\end{pmatrix}$$

In Matlab we can find null space : $\underline{N} = \text{null}(\underline{B})$

$$\underline{N} = \text{spnull}(\underline{B})$$

However this is very slow for large systems.

It turns out we can find null space easily because our constraints are simple (involve only one unknown)

BC sets $h_8 = 0$

$$\underline{\underline{I}} = \begin{bmatrix}1&&&&&&&\\&1&&&&&&\\&&1&&&&&\\&&&1&&&&\\&&&&1&&&\\&&&&&1&&\\&&&&&&1&\\&&&&&&&1\end{bmatrix}$$

$\underline{B} = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1]$

The remaining unknowns

$h_1 - h_7$ = null space of $\underline{B}$

$\Rightarrow$ basis for $N(\underline{B})$ is simply the remaining colums of $\underline{\underline{I}}$

$$\underline{\underline{N}} = \begin{bmatrix}1&&&&&&\\&1&&&&&\\&&1&&&&\\&&&1&&&\\&&&&1&&\\&&&&&1&\\&&&&&&1\\&&&&&&\end{bmatrix}$$

Notes on implementation: build_bnd_m

dof_dir = Grid.dof_xmax ;

% Build $\underline{\underline{B}}$ from $\underline{\underline{I}}$

$\underline{\underline{B}}$ = I(dof_dir, :) ;

% build $\underline{\underline{N}}$ from $\underline{\underline{I}}$

$\underline{\underline{N}}$ = $\underline{\underline{I}}$; $\underline{\underline{N}}$(:, dof_dir) = [ ] ;

Essentially we split $\underline{\underline{I}} \begin{smallmatrix} \nearrow \underline{\underline{B}} \\ \searrow \underline{\underline{N}} \end{smallmatrix}$