Lecture 7: Heterogeneous BC's

Logistics:
- record todays class ? ✓
- HW1 grades on canvas
- HW2 due Thursday

Last time:
- Scaling analysis of steady linear confined aquifer
  → eliminated all 5 parameters

PDE: $-\frac{d^2h'}{dx'^2} = 1 \qquad x' \in [0,1]$

BC: $\left.\frac{dh'}{dx'}\right|_0 = 0 \qquad h'(1) = 0$

$\Rightarrow \; h_c = \frac{f_s L^2}{bK} \sim \Delta h$

$\Big\} \; h' = \frac{1}{2}(1 - x'^2)$

- Homogeneous Dirichlet BC's

reduced system: $\underline{\underline{L}}_r \underline{h}_r = \underline{f}_{sr}$

projection matrix $\underline{\underline{N}} = \text{null}(\underline{\underline{B}})$

$\underline{h} = \underline{\underline{N}} \, \underline{h}_r \qquad \underline{h}_r = \underline{\underline{N}}^T \underline{h}$

$\underline{\underline{L}}_r = \underline{\underline{N}}^T \underline{\underline{L}} \, \underline{\underline{N}}$

Today:
- heterogeneous constraints
- layered media

Heterogeneous Dirichlet BC's

Impose analytic solution at last cell center as BC

$\Rightarrow$ not zero

PDE : $-\frac{d^2h'}{dx'^2} = 1 \qquad x' \in [0,1]$

BC : $\left.\frac{dh'}{dx'}\right|_0 = 0 \qquad h'(1) = 0 \qquad h'(x_{c,Nx}) = h_{ana}(x_{c,Nx})$
$= g$

Linear system for BC: $\underline{\underline{B}}\,\underline{h} = g$
↑
same as before

Because system is linear we can split $\underline{h} = \underline{h}_0 + \underline{h}_p$ into homogeneous soln h_0 and a particular soln h_p.

homog. soln: $\underline{\underline{B}}\,\underline{h}_0 = \underline{0}$
particular soln: $\underline{\underline{B}}\,\underline{h}_p = g$

$\Big\}\ \underline{\underline{B}}(\underbrace{\underline{h}_0 + \underline{h}_p}_{\underline{h}}) = g$

Note: $\underline{h}$ is unique

But the split of $\underline{h}$ into $\underline{h}_0$ and $\underline{h}_p$ is not unique, but there is an simplest choice

Two questions: 1) how do we find suitable $\underline{h}_p$?

→ have a choice

2) Given $\underline{h}_p$ what is associated $\underline{h}_o$?

Start with 2: Suppose we have $\underline{h}_p$

$$\underline{\underline{L}}\,\underline{h} = \underline{f}_s \qquad \underline{h} = \underline{h}_o + \underline{h}_p$$

$$\underline{\underline{L}}(\underline{h}_o + \underline{h}_p) = \underline{f}_s$$

$$\underline{\underline{L}}\,\underline{h}_o = \underline{f}_s - \underline{\underline{L}}\,\underline{h}_p = \underline{f}_s + \underline{f}_D \qquad \underline{f}_D = -\underline{\underline{L}}\,\underline{h}_p$$

$$= \underline{f} \qquad \underline{f} = \underline{f}_s + \underline{f}_D$$

We have:

$$\left.\begin{aligned} \underline{\underline{L}}\,\underline{h}_o &= \underline{f} \\ \underline{\underline{B}}\,\underline{h}_o &= \underline{0} \end{aligned}\right\} \text{ homog. problem}$$

To solve we project into null space of B

$$\boxed{\underline{\underline{L}}_r\,\underline{h}_{or} = \underline{f}_r} \qquad \underline{\underline{L}}_r = \underline{\underline{N}}^T \underline{\underline{L}}\,\underline{\underline{N}}$$

$$\underline{h}_{or} = \underline{\underline{N}}^T \underline{h} \qquad \underline{f}_r = \underline{\underline{N}}^T \underline{f}$$

$$\Rightarrow \quad \underline{h}_o = \underline{\underline{N}}\,\underline{h}_{or} \qquad \Rightarrow \quad \underline{h} = \underline{h}_o + \underline{h}_p$$

Find a particular solution $\underline{h}_p$:

Note that $\underline{h}_p$ does <u>not</u> need to satisfy $\underline{\underline{L}}\,\underline{h}_p \neq f$

that is taken care of by homog. solution.

Hence $\underline{h}_p$ simply needs to solve $\underline{\underline{B}}\,\underline{h}_p = g$

Intuition: Given that $\underline{\underline{B}}$ is composed Nc rows of $\underline{\underline{I}}$, $\underline{h}_p$ needs to have Nc entries of g in the right places. $\underline{h}_p = \underline{\underline{B}}^T g$

(This would be sufficient for us)

To solve more general constraint problem we need to solve $\underline{\underline{B}}\,\underline{h}_p = g$ (not square)

$\Rightarrow$ need to make it square

most obvious: $\underbrace{\underline{\underline{B}}^T \underline{\underline{B}}}_{N_x \cdot N_x} \underline{h}_p = \underline{\underline{B}}^T g$ — $\underline{\underline{B}}^T\underline{\underline{B}}$ is not invertible because it has at most N_c^2 non-ero entries

$N_c \cdot N_x$

Instead we want to condense this to $N_c \cdot N_c$ system.

Want to solve reduced system: $\underline{B}_r \, \underline{h}_{pr} = g$

$N_c \cdot N_c \quad N_c \cdot 1 \quad N_c \cdot 1$

$\underline{h}_{pr} = \underline{\underline{B}} \, \underline{h}_p$

$N_c \cdot 1 \quad N_c \cdot N_x \quad N_x \cdot 1$

from eqn: $\underline{\underline{B}} \, \underline{h}_p = g$

compare $\Rightarrow$ $\underline{h}_{pr} = g$

$\underline{h}_p = \underline{\underline{B}}^T \underline{h}_{pr}$ substitute this into eqn

$\underline{\underline{B}} \, \underline{h}_p = g$ $\quad \boxed{\underline{\underline{B}}\,\underline{\underline{B}}^T \underline{h}_{pr} = g}$ $\quad \underline{h}_p = \underline{\underline{B}}^T \underline{h}_{pr}$

$N_c \cdot N_c \quad N_c \cdot 1 \quad N_c \cdot 1$

Summary: Solving linear system with constraints.

PDE: $\underline{\underline{L}} \, \underline{h} = \underline{f}_s$

BC: $\underline{\underline{B}} \, \underline{h} = g$

Step 1: find particular solution

$\underline{h}_p = \underline{\underline{B}}^T \underline{h}_{pr} \qquad \underline{h}_{pr} = \underline{\underline{B}}\,\underline{\underline{B}}^T \setminus g$

$$\boxed{\underline{h}_p = \underline{\underline{B}}^T (\underline{\underline{B}}\,\underline{\underline{B}}^T \setminus g)}$$

Step 2: find associated hom. solution

$$\underbrace{\underline{\underline{N}}^T \underline{\underline{L}}\,\underline{\underline{N}}}_{\underline{\underline{L}}_r} \; \underbrace{\underline{\underline{N}}^T h_0}_{h_{0r}} = \underline{\underline{N}}^T (f_s - \underline{\underline{L}}\, h_p)$$

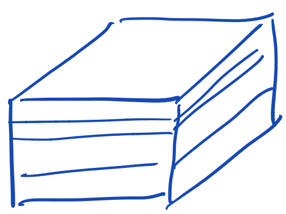
$$\boxed{\underline{h}_0 = \underline{\underline{N}} \left[(\underline{\underline{N}}^T \underline{\underline{L}}\, \underline{\underline{N}}) \backslash (\underline{\underline{N}}^T (\underline{f}_s - \underline{\underline{L}}\, \underline{h}_p)) \right]}$$

Step 3: $\underline{h} = \underline{h}_0 + \underline{h}_p$

All of this will be encapsulated in a general purpose function that solves linear boundary value problems (LBVP)

$\Rightarrow$ solve_lbvp($\underline{\underline{L}}, \underline{f}_s, \underline{\underline{B}}, \underline{g}, \underline{\underline{N}}$)

Effective Properties of Layered Materials



Stack of N layers with thickness Δl_i conductivity K_i $i \in [1, .., N]$

How does this affect flow?

$\Rightarrow$ 3D problem - computationally

We can analyze 2 limiting cases

1) Flow is perpendicular to layers (in-series)

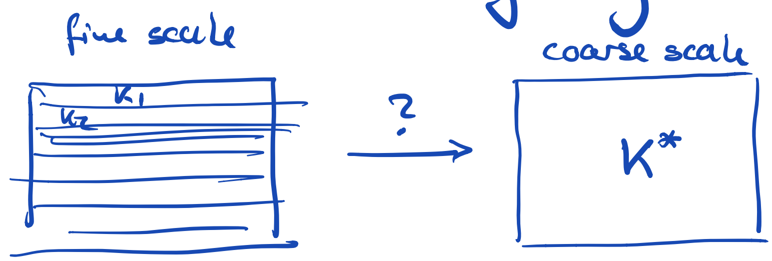
2) Flow is parallel to layers (in-parallel)

$\Rightarrow$ In these limits flow is 1D

Try and understand the effect of layering by

by finding an effective property that describes

the entire stack of layers.



Fine scale: layered medium k changes with location (heterog.)

Coarse scale: homogeneous medium, but K^* will

depend on direction. (anisotropy)