Lecture 8: Layered media & variable coef.

Logistics: - HW 2 due ✓

- (HW 3 posted)

Last time: Heterogeneous BC's

$$\underline{\underline{B}}\,\underline{h} = g$$

h

$$\underline{h} = \underline{h}_0 + \underline{h}_p$$

$q \cdot n \sim T$

I) $\underline{\underline{B}}\,\underline{\underline{B}}^T\,\underline{h}_{pr} = g$

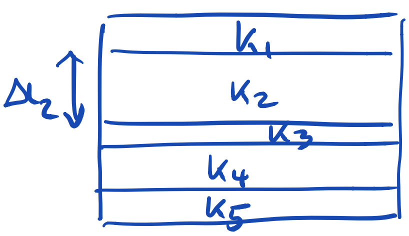
II) $\underline{\underline{L}}\,\underline{h}_0 = f_s - \underline{\underline{L}}\,\underline{h}_p$

~~$h = h_0 + h$~~

- Effective properties of layered media

Today: - Layered media
- effective properties
- anisotropy (dependence on dir.)
- mean crustal permeability
- variable coefficients

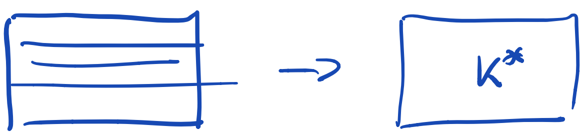
Layered media



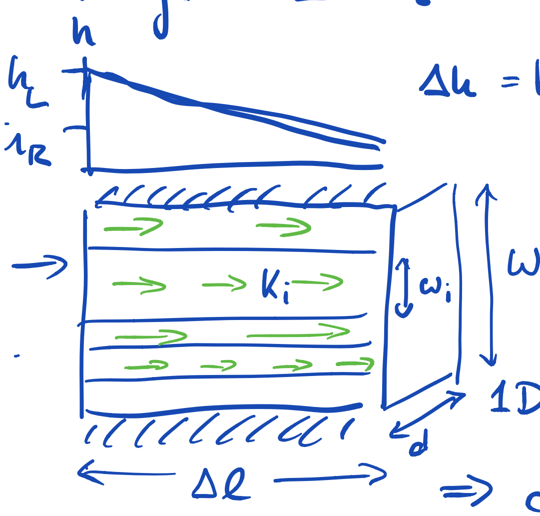
stack of N layers

$K_i \quad \Delta l_i$

1) Flow along layers



Can we find K^* that gives the same Q for Δh ?



$\Delta h = h_R - h_L$

every layer experiences same Δh

1D flow along each layer

$\Rightarrow$ consider them separately

Darcy ith layer: $\quad Q_i = - \underbrace{d\, w_i}_{A_i} \, k_i \, \frac{\Delta h}{\Delta l}$

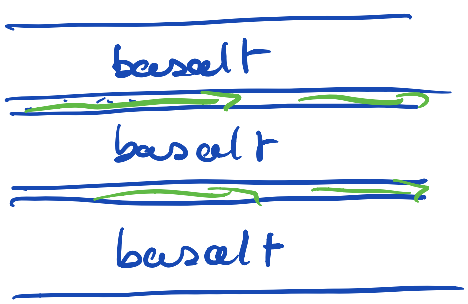
Darcy for stack: $Q = -d\,W\,K^*_{\parallel}\,\frac{\Delta h}{\Delta \ell}$

Total flow $Q = \sum_{i=1}^{N} Q_i$

$$-\cancel{d}\,W\,K^*_{\parallel}\,\cancel{\frac{\Delta h}{\Delta L}} = \sum_{i} -\cancel{d}\,w_i\,K_i\,\cancel{\frac{\Delta h}{\Delta L}}$$

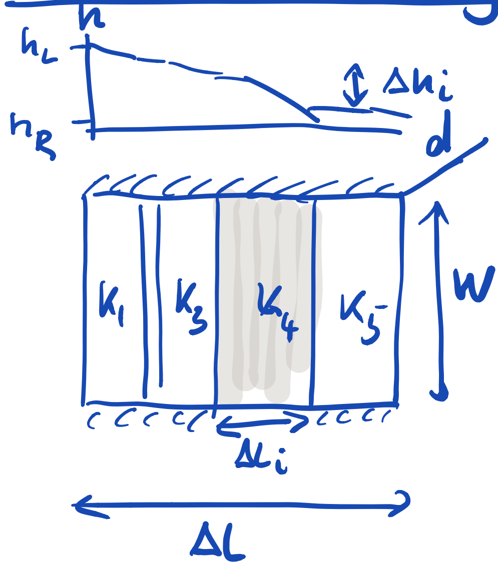
$$\boxed{K^*_{\parallel} = \sum_{i=1}^{N} \frac{w_i}{W} K_i}$$

Effective hyd. cond. for flow along layers is weighted <u>arithmetic</u> average.



high K layers "dominate"

Flow across layers



1D flow perpendicular to layers.

$Q_i = Q$ $\quad A = dW = \text{const}$

$\Rightarrow \quad q_i = q$

Darcy i-th: $q = -K_i \dfrac{\Delta h_i}{\Delta l_i}$

Darcy stack: $q = -K_\perp^* \dfrac{\Delta h}{\Delta L}$

$\Delta h = \sum_{i=1}^{N} \Delta h_i \qquad \Delta h_i = -q \dfrac{\Delta l_i}{K_i} \qquad \Delta l = \sum \Delta l_i$

$$K_\perp^* = -q \frac{\Delta L}{\Delta h} = -\cancel{q} \frac{\Delta L}{-\cancel{q} \sum \frac{\Delta l_i}{K_i}} = \frac{\Delta L}{\sum \frac{\Delta l_i}{K_i}} = \frac{1}{\sum_{i=1}^{N} \frac{\Delta l_i / \Delta L}{K_i}}$$

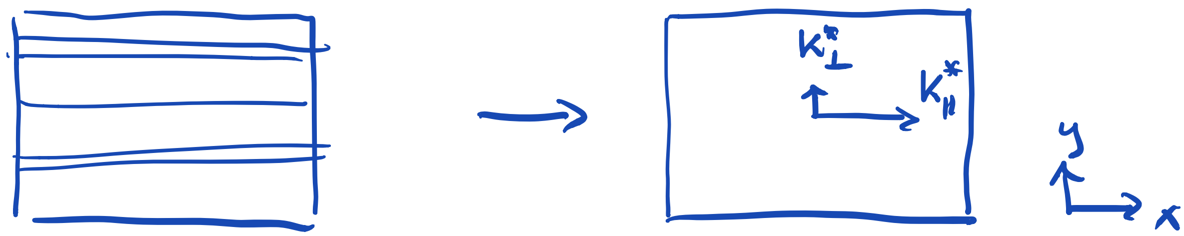
$$\boxed{K_\perp^* = \frac{1}{\sum_{i=1}^{N} \frac{\Delta l_i / \Delta L}{K_i}}} \qquad K_\perp^* \le K_\parallel^*$$

Effective hydr. cond. for flow across layers is weighted harmonic average

$\Rightarrow$ lowest K_i will dominate

Fine scale structure → coarse anisotropy

fine scale | coarse scale



heterogeneaus
isotropic

$K = K(\underline{x})$

homogeneous
anisotropic

$\underline{\underline{K}} = \begin{pmatrix} K_\parallel^* & 0 \\ 0 & K_\perp^* \end{pmatrix}$

Variable coefficeuts

Heterogeneity is key element of porous media

$K > 0$

Continuous Equ: $-\nabla \cdot [\, K(\underline{x}) \, \nabla h \,] = f_s$

Discrete Equ: $-\underline{\underline{D}} * [\, \underline{\underline{Kd}} * \underline{\underline{G}}\, \underline{h} \,] = \underline{f}_s$

Size of $\underline{\underline{Kd}}$?

D	Kd	G
$Nx \cdot (Nx+1)$	$(Nx+1) \cdot (Nx+1)$	$(Nx+1) \cdot Nx$

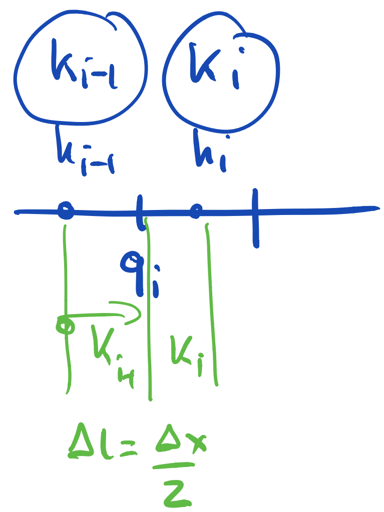
$\Rightarrow$ $\underline{\underline{Kd}}$ is $Nx+1$ by $Nx+1$ matrix associated with faces

$\Rightarrow$ mean K on each face

homogeneous case: $\underline{\underline{Kd}} = K \, \underline{\underline{I}}$

Darcy: $q = - \underline{\underline{Kd}} \; \underbrace{\underline{\underline{G}} \, \underline{h}}_{dh}$

mean



$$q_i = - K_{i-\frac{1}{2}} \frac{h_i - h_{i-1}}{\Delta x}$$

$K_{i-\frac{1}{2}}$ = mean K_i K_{i-1}

$\Rightarrow$ harmonic mean because flow across layers

$$\frac{\frac{\Delta x}{2}}{\Delta x}$$

$$K_{i-\frac{1}{2}} = \frac{2}{\frac{1}{K_{i-1}} + \frac{1}{K_i}} \qquad \Delta l_i / \Delta l = \frac{1}{2}$$

$\underline{\underline{Kd}}$ is a diagonal matrix with the harmonic average of K to each face.

Computing means

comp_mean_matrix.m

→ compute both harmonic & arithmetic means of cell center quantities on faces

1) Generate mean matrix $\underline{\underline{M}}$ so that

$\underline{Kmean} = \underline{\underline{M}} * \underline{K}$ comp. arithm. mean via matrix-vector product

Note: $\underline{\underline{M}}$ has same fill pattern as $\underline{\underline{G}}$

$$\boxed{\underline{\underline{M}} = \frac{\Delta x}{2} \, |\underline{\underline{G}}|}$$

beware typo in notes

Note: zero on bnd

2) Harmonic average

$$\underline{Kmean} = 1./(\underbrace{\underline{\underline{M}} * (1./\underline{K})}_{0 \text{ on bnd}})$$

need to set $\underline{Kmean}$(dof_bnd) = 0.

Place Kmean on diagonal of Kd