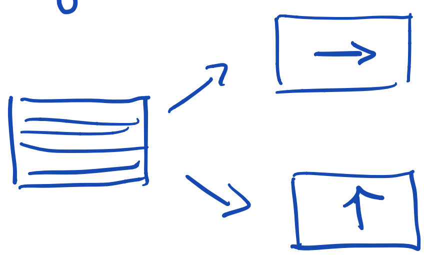
Lecture 9: Fluxes and Neumann BC

Logistics: - HW 3 postponed until March 4th

- please complete HW2 (added exercise)

Last time: - Layered media



$K^*_{\parallel} = \sum_{i=1}^{N} \frac{w_i}{W} k_i$ arith

$K^*_{\perp} = \frac{\ell}{\sum_i \frac{\ell_i}{k_i}}$ harm

$\Rightarrow$ anisotropy (dependence on direction) tensor

- Variable coefficients

$$-\nabla \cdot [K(\underline{x}) \nabla h] = f_s$$

$$-\underline{\underline{D}} * \underline{\underline{Kd}} * \underline{\underline{G}} * \underline{h} = \underline{f_s}$$

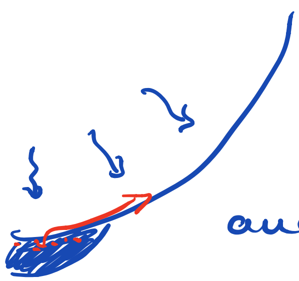
harmonic ave.

- Mean matrix: Kmean $= \underline{\underline{M}} * \underline{K}$ arith

harm Kmean $=$ 1./(M*(1./K))

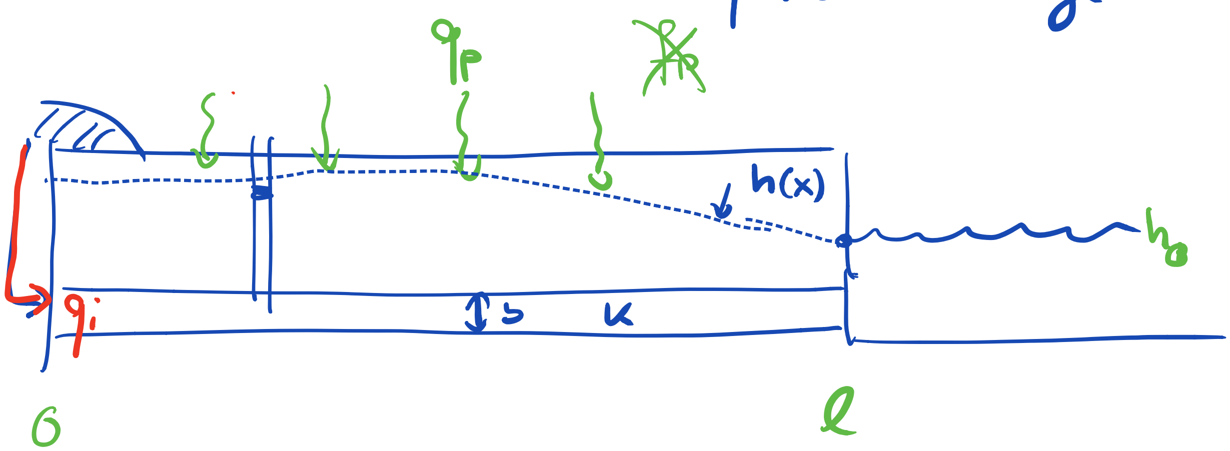
Today: • New example problem
• Neumann BC
• Flux computations

Example 2: Aquifer with polar recharge



Clifford and Parker (2001)
Ice caps act as insulators and ~~the~~ heat flow from interior will lead to basal melting of ice cap.

$\Rightarrow$ two fluid sources: • precipitation
• polar recharge



PDE: $-\frac{d}{dx}\left[bK\frac{dh}{dx}\right] = q_b \qquad x \in [0, \ell]$

BC: $q_i = -K\frac{dh}{dx}\Big|_0 \Rightarrow \frac{dh}{dx}\Big|_0 = -\frac{q_i}{K}$ Neumann

$h(\ell) = h_0$

Non-dimensionalize:

$x' = \frac{x}{\ell} \qquad h' = \frac{h-h_0}{h_c}$

PDE: $-\frac{d^2h'}{dx'^2} = \frac{q_p\ell^2}{bKh_c^{I}} = 1 \qquad x' \in [0, 1]$

BC: $\frac{dh'}{dx'}\Big|_0 = -\frac{q_i\ell}{Kh_c^{I}} = \Pi$

$\Rightarrow$ Two dimensionless groups

I: $\frac{q_p\ell^2}{bKh_c} = 1 \Rightarrow h_c^{I} = \frac{q_p\ell^2}{Kb}$

II: $\frac{q_i\ell}{Kh_c} = 1 \Rightarrow h_c^{II} = \frac{q_i\ell}{K}$

choose the h_c associated with dominant process → precipitation

$$h_c = h_c^F = \frac{q_p \ell^2}{K b}$$

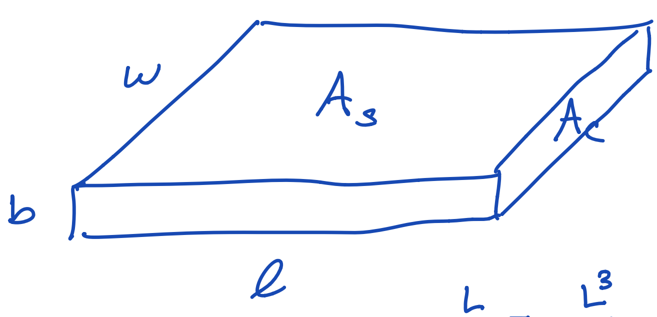
⇒ dimensionless governing parameter

$$\Pi = \frac{q_i \ell}{K h_c} = \frac{q_i b}{q_p \ell}$$

Dim. Equations:

$$\boxed{\begin{array}{lll} \text{PDE:} & -\dfrac{d^2 h'}{dx'^2} = 1 & x' \in [0,1] \\[2ex] \text{BC:} & \left.\dfrac{dh'}{dx'}\right|_0 = \Pi & h'(1) = 0 \end{array}}$$

Interpretation of Π:



surface area

$A_s = \ell w$

x-section area

$A_c = b w$

$$\frac{L}{T} = \frac{L^3}{LT}$$

$$Q_p = A_s q_p = \ell w q_p$$

$$Q_i = A_e q_i = b\omega q_i$$

$$\Pi = \frac{q_i b}{q_p l} = \frac{q_i b \omega}{q_p l \omega} = \frac{q_i A_c}{q_p A_s} = \frac{Q_i}{Q_p}$$

Analytic solution:

integrate once: $-\frac{dh'}{dx'} = x' + c_1$

use Neuman BC: $c_1 = \Pi$

Integrate again: $-h' = \frac{x'^2}{2} + \Pi + c_2$

use Dirchlet BC: $c_2 = -\frac{1}{2} - \Pi$

$$\boxed{\begin{aligned} h' &= \tfrac{1}{2}(1 - x'^2) + \Pi(1 - x') \\ q' &= x' + \Pi \end{aligned}}$$

note $q' = -\frac{dh'}{dx}$

Neumann BC's

Neumann BC do not prescribe unknown.

$\Rightarrow$ cannot implement as constraints!

Example 2: $q|_0 = -\frac{dh}{dx}\Big|_0 = \Pi$

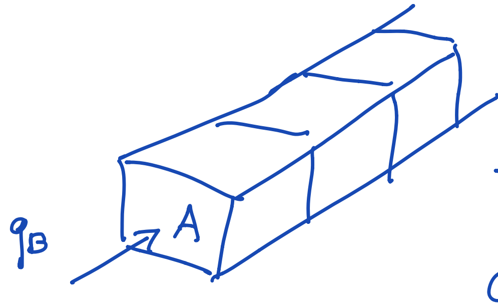
Note: In our implementation we want inflow to be positive

$q \cdot \hat{n}_i = q_B = \Pi$ $\qquad q_B > 0$ inflow

$\hat{n}_i$: inner normal

Implementation of Neumann BC

We implement Neumann BC as an equivalent source/sink term to ensure discrete mass conservation.



Total flow rate across bnd:

$Q_b = A\, q_B$

Equivalent source term: $Q_b = V f_n$

f_n is Neumann source term cell volume

$$\boxed{f_n = q_b \frac{A}{V}}$$ (for single cell)

In general f_n is N_x by 1 is r.h.s. vector with N_n non-zero entries, one for each cell with Neumann BC.

For a problem with Neuman BC the linear system is:

$$\underline{\underline{L}}\,\underline{h} = \underline{f_s} + \underline{f_n}$$

Matlab implementation:

Three vectors:

BC.dof_neu = Nn by 1 vector cells with Neu BC

BC.dof_f_neu = Nn by 1 vector faces with Neu BC

BC.qb = Nn by 1 vector of fluxes

Grid.A = Nf by 1 $\qquad$ Nf = Nx+1

Grid.V = N by 1 $\qquad$ N = Nx

Compute and place Nn entries of fn:

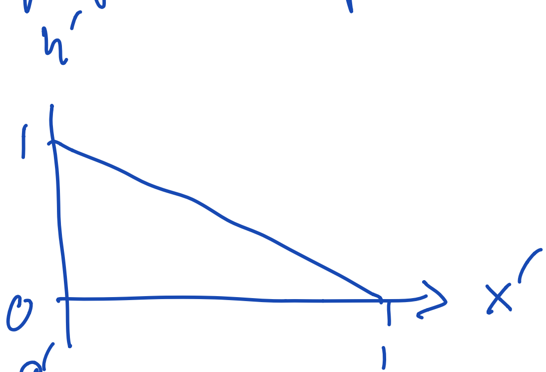
fn(BC.dof_neu) = BC.qb * Grid.A(BC.dof_f_neu)/Grid.V

(dof_neu)

BC.

⇒ added to build_bnd.m

Aquifer with polar recharge no precip.



BC $\frac{dh'}{dx'} = -\frac{q_1 \ell}{K h_c}$

$q' = -\frac{dh'}{dx'}$