Lecture 3: Introduction to numerics   Jan 26

Logistics: none

Last time: - General balance law

$$\frac{\partial u}{\partial t} + \nabla \cdot \underline{j} = \hat{f}_s \qquad u = \text{unknown}$$

- Fluid mass conservation $\qquad u = \phi\rho$

$$\cancel{\frac{\partial}{\partial t}(\phi\rho)} + \nabla\cdot(\rho \underline{q}) = \rho f_s$$

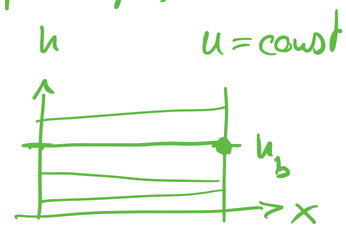
$$\underline{q} = -K\,\underline{\nabla h}$$

- Incompressible flow $\quad \rho = \text{const.}$

$$\boxed{-\nabla\cdot[K\nabla h] = f_s}$$

Poisson Eqn
(Elliptic Eqn)

- Boundary conditions



• Dirichlet: $h = \underline{h_b}$

• Neumann: $\underline{q}\cdot\hat{n} = -\underline{q_b}$

$q = 0$

$f_s = 0$

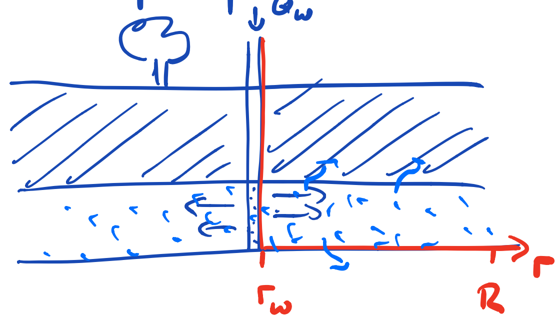
Today: - Introduction to numerics

Motivate our approach

- Review FD

- Example problem

Example problem: Flow around an injection well



$\nabla\cdot q = f_s$

$-\nabla\cdot k\nabla h = 0$

$\nabla\cdot(\circ) = \frac{1}{r}\frac{d}{dr}(r\ \circ)$

$\nabla = \frac{d}{dr}$

PDE: $-\frac{1}{\cancel{r}}\frac{d}{dr}\left(r\frac{dh}{dr}\right) = 0$ on $r\in[r_w\ R]$

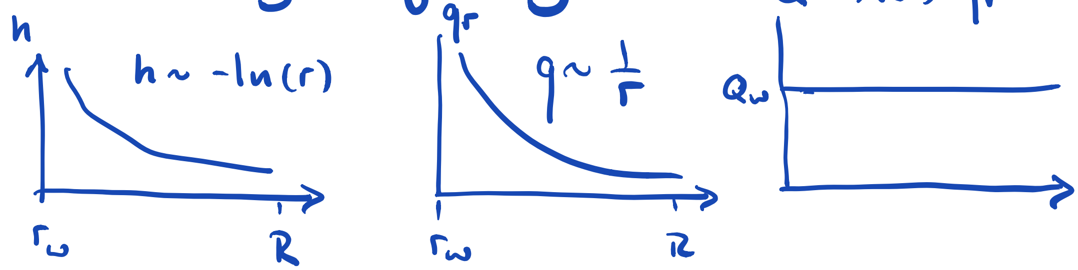
BC: $Q_w = A_w q_r(r_w) = -A_w k\frac{dh}{dr}\Big|_{r_w}$ (Neumann) $A_w = 2\pi r_w H$

$h(r=R) = h_B$

$\frac{dh}{dr}\Big|_{r_w} = -\frac{Q_w}{A_w k}$

$\Rightarrow$ solve by integrating twice

$Q = \overset{2\pi r H}{A(r)}\, q_r$



Finite difference discretization

PDE: $\frac{d}{dr}\left(r\frac{dh}{dr}\right) =$

$= r\frac{d^2h}{dr^2} + 1\frac{dh}{dr}$

$\frac{dh}{dr}\Big|_i = \frac{h_{i+1} - h_{i-1}}{2\Delta r} = \underline{\underline{D}}$

$\frac{d^2h}{dr^2}\Big|_i = \frac{h_{i+1} - 2h_i + h_{i-1}}{\Delta r^2} = \underline{\underline{D}}\cdot\underline{\underline{D}}$

Observations: 1) Surprisingly large errors

2) Mass is not conserved

For practical problems we need an approach that will conserve mass exactly even on a finite grid. $\Rightarrow$ $\boxed{\text{Discrete conservation}}$

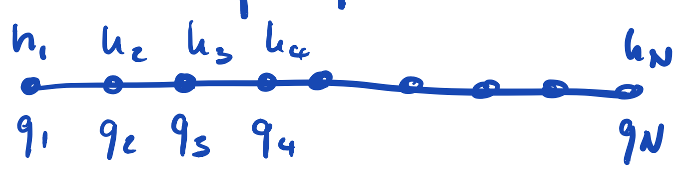
Finite differencing in conservation form

Discretize with divergence intact.

mass bal: $\nabla \cdot q = f_s$ $\qquad \cancel{\frac{1}{r}} \frac{d}{dr}(r\, q_r) = 0$

Darcy law: $q = -K \nabla h$ $\qquad q_r = -K \frac{dh}{dr}$

Introduce $q_r \hat{=} q$ as additional unknown



colocated grid

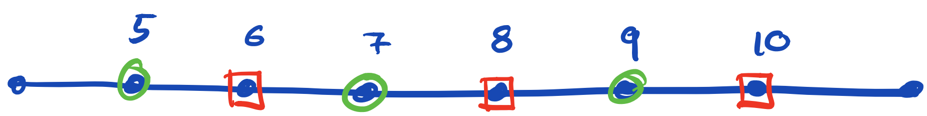
$$\frac{d}{dr}(r\,q) \approx \frac{r_{i+1} q_{i+1} - r_{i-1} q_{i-1}}{2\Delta r}$$

$$q_i = -K \frac{dh}{dr} \approx -K_i \frac{h_{i+1} - h_{i-1}}{2\Delta r}$$

central differences

$$\frac{1}{2\Delta r}\left(r_{i+1}\underbrace{\left(-K_{i+1}\frac{h_{i+2}-h_i}{2\Delta r}\right)}_{q_{i+1}} - r_{i-1}\underbrace{\left(-K_{i-1}\frac{h_i-h_{i-2}}{2\Delta r}\right)}_{q_{i-1}}\right)$$

$$-\frac{1}{4\Delta r^2}\left(r_{i+1}K_{i+1}h_{i+2} - (r_{i+1}K_{i+1}+r_{i-1}K_{i-1})\,h_i + r_{i-1}K_{i-1}h_{i-2}\right)$$



$i=8 \rightarrow \quad h_{8+2}=h_{10} \quad h_8 \quad h_{8-2}=h_6$

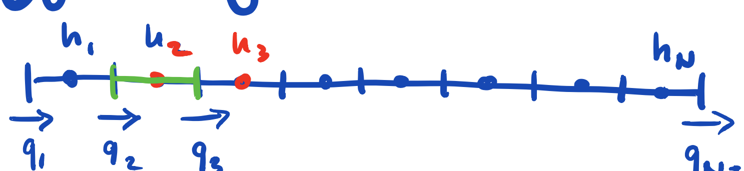
$i=7 \rightarrow \quad h_9 \quad h_7 \quad h_5$

$$\underline{h} = \begin{bmatrix} h_1 \\ h_3 \\ h_5 \\ \vdots \\ h_2 \\ h_4 \\ h_6 \\ \vdots \end{bmatrix} \qquad \underline{\underline{L}} = \begin{bmatrix} \boxed{\text{odd}} & \\ & \boxed{\text{even}} \end{bmatrix}$$

$\Rightarrow$ decoupling between even and odd nodes

The way to resolve the decoupling

$\Rightarrow$ staggered grid



$q_3 = -k\,\frac{h_3-h_2}{\Delta x}$

$\nabla\cdot q|_2 = \frac{q_3-q_2}{\Delta x} = f_2$