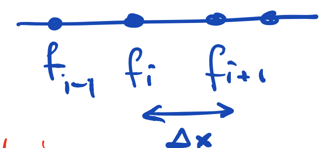
Lecture 4: Discrete Operators

Logistics: - HW 1 is due Th Feb 4, 9:30 am

- get started before office hrs on Mon so we can sort out problems

Last time: Introduction to numerics

- Finite differences



$$\left.\frac{df}{dx}\right|_{x_i} \approx \frac{f_{i+1} - f_{i-1}}{2\Delta x}$$

$\Rightarrow$ differentiation matrix

- Example problem: Flow around well

$$\frac{1}{r}\frac{d}{dr}\left(r\,\frac{dh}{dr}\right) = 0 \qquad r\frac{d^2h}{dr^2} + \frac{dh}{dr} = 0 \qquad \left.\frac{dh}{dr}\right|_{r_w} \sim Q_w$$

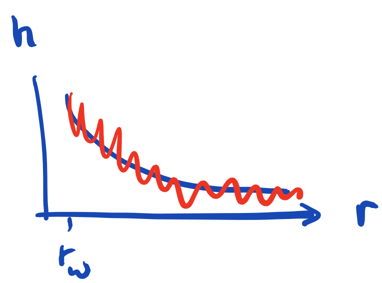
$\Rightarrow$ bad approximation

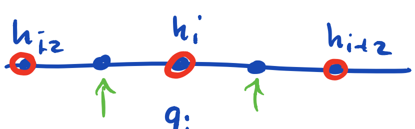
$\Rightarrow$ mass is not conserved

- Discrete conservation

$$\nabla \cdot q = f_s \qquad \frac{dq}{dx}$$

$$q = -k\nabla h \qquad \frac{dh}{dx}$$



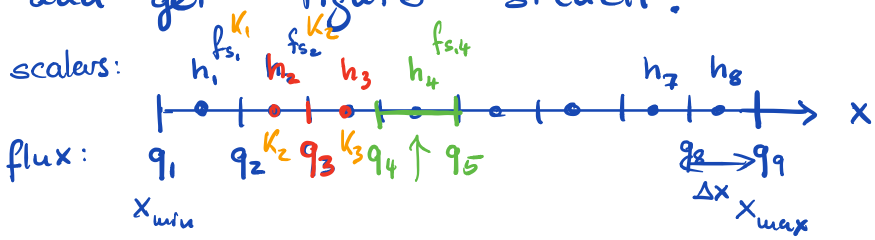


$\Rightarrow$ even-odd decoupling $\Rightarrow$ oscillations

Today:
- Staggered grid ✓
- Conservative finite differences ✓
- Discrete operators ✓
- coding 'basics'

Staggered grid

Reason is avoid the decoupling of even-odd nodes and get "tighter" stencil.



$x \in [x_{min}\ x_{max}]$

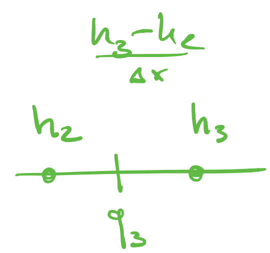
$Lx = x_{max} - x_{min}$

$Nx = 8$ cells

$\Delta x = \frac{Lx}{Nx}$

head is approximated at cell center (cell = control volume)

fluxes is approximated at cell faces.



Discretize div-grad system:

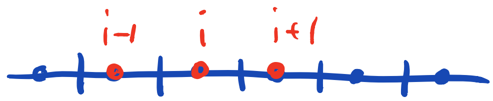
1) $\nabla \cdot q = f_s \xrightarrow{1D} \frac{dq}{dx} = f_s \xrightarrow{FD} \frac{q_{i+1} - q_i}{\Delta x} = f_{s,i}$ (central diff relative to cell center)

2) $q = -K \nabla h \xrightarrow{1D} q = -K \frac{dh}{dx} \xrightarrow{FD} q_i = -K_{i-\frac{1}{2}} \frac{h_i - h_{i-1}}{\Delta x}$

Subsitute q_i into mass balance

$$-\frac{1}{\Delta x^2}\left[\underbrace{K_{i+\frac{1}{2}}(h_{i+1}-h_i)}_{q_{i+1}} - \underbrace{K_{i-\frac{1}{2}}(h_i-h_{i-1})}_{q_i}\right] = f_{s,i}$$

$$-\frac{1}{\Delta x^2}\left[K_{i+\frac{1}{2}}h_{i+1} - (K_{i+\frac{1}{2}}+K_{i-\frac{1}{2}})h_i + K_{i-\frac{1}{2}}h_{i-1}\right] = f_{s,i}$$



narrow / tight stencil

Discrete operators

$$\nabla \cdot q = f_s \longrightarrow \underline{\underline{D}} * \underline{q} = \underline{f_s}$$

$$q = -K\nabla h \longrightarrow \underline{q} = -K\,\underline{\underline{G}} * \underline{h}$$

Discrete Gradient

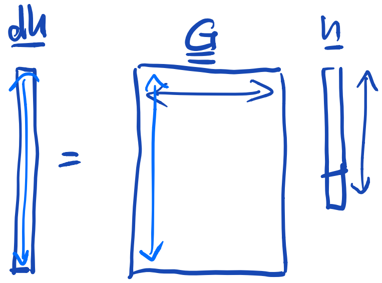
$q \sim \frac{dh}{dx} = \nabla h$ Gradient takes a scaler (h) from cell center and returns a derivative/flux on cell faces

$\underline{xc}$ = vector of cell center location Nx by 1

$\underline{xf}$ = vector of cell face locations Nx+1 by 1

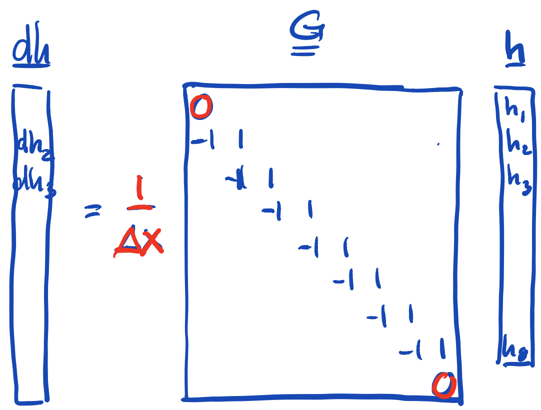
$\underline{h} = h(\underline{xc})$ $\qquad \underline{dh} = \left.\frac{dh}{dx}\right|_{\underline{xf}}$ $\qquad \underline{dh} = \underline{\underline{G}} * \underline{h}$

$Nx{+}1 \cdot 1 \; \{ \; Nx{+}1 \cdot Nx \qquad Nx \cdot 1$

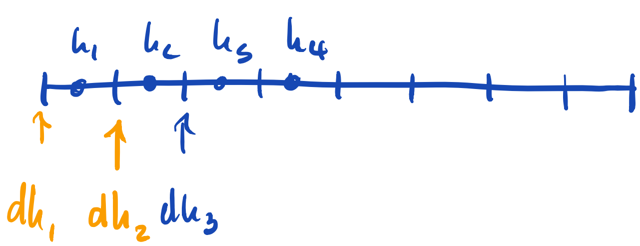


$\Rightarrow \underline{\underline{G}}$ is $Nx+1$ by Nx matrix

$Nx = 8$



build in a zero gradient at boundary



dh_1 ?

$$dh_2 = \frac{h_2 - h_1}{\Delta x} \qquad dh_3 = \frac{h_3 - h_2}{\Delta x}$$

Divergence operator

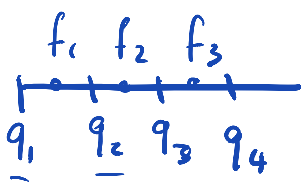
$\nabla \cdot q = f_s \qquad f_s = \underline{\underline{D}} * \underline{q} = \underline{f_s}$

$Nx \cdot Nx{+}1 \; \{ \; Nx{+}1 \cdot 1 \qquad Nx \cdot 1$

$N_x = 8$

$$\underset{=}{f_s} = \frac{1}{\Delta x}\,\underset{=}{D}\,\underset{=}{q}$$

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ - \\ \\ \\ \\ f_8 \end{bmatrix} = \frac{1}{\Delta x} \begin{bmatrix} -1 & 1 & & & & & & & \\ & -1 & 1 & & & & & & \\ & & -1 & 1 & & & & & \\ & & & -1 & 1 & & & & \\ & & & & -1 & 1 & & & \\ & & & & & -1 & 1 & & \\ & & & & & & -1 & 1 & \\ & & & & & & & -1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \\ \\ \\ \\ \\ q_9 \end{bmatrix}$$



$$\frac{dq}{dx} = f_s \qquad \frac{q_{i+1} - q_i}{\Delta x} = f_i$$

$$\frac{q_2 - q_1}{\Delta x} = f_i$$

Relationship between $\underset{=}{D}$ and $\underset{=}{G}$

$$\boxed{\underset{=}{G} = -\underset{=}{D}^T} \qquad \text{interior}$$

then we need to zero out bounderies

Laplacian operator

$$\nabla^2 = \nabla \cdot \nabla \qquad \left(-\nabla \cdot K \nabla h = f_s \right)$$

$$\underset{=}{L} = \underset{=}{D} * \underset{=}{G} \qquad -\underset{\substack{\uparrow \\ \text{const}}}{K}\, \nabla^2 h = f_s$$