Lecture 2: Balance laws

Jan 21, 2021

Logistics: - Office hours Tu 12-2pm

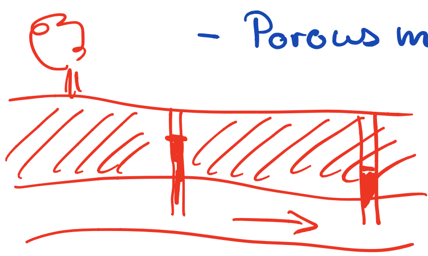
Mo 9-10 am same Zoom link

Fr 4:30-6pm

- HW 1 posted on Matlab Grader due Feb 4th

Last time: - Course logistics

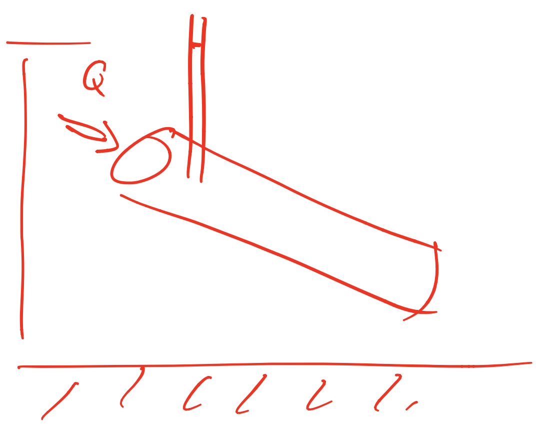
- Course project: ground water flow into impact craters on Mars



- Porous media: - "Darcy scale" vs "Pore scale"

- saturated vs. unsaturated

- volume fractions, saturations



- Darcy's law: $\boxed{q = -\underline{\underline{K}} \nabla h}$

q = vol. flux $\left[\frac{L^3}{L^2 T}\right]$ (specific discharge)

h = hydraulic head $[L]$

$\underline{\underline{K}}$ = hydraulic conductivity

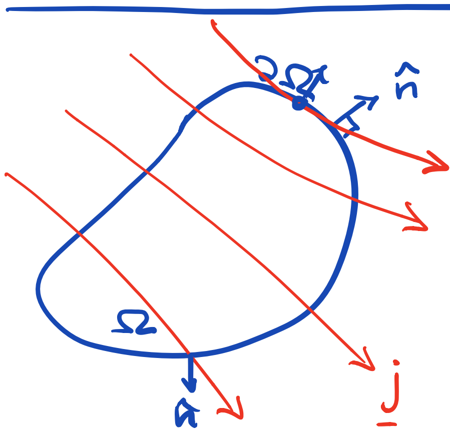
Today: • Derivation of general balance law

• Fluid mass balance

• Incompressible flow

$\Rightarrow$ Governing eqn for steady saturated flow

General balance law



Account for change of $u(\underline{x},t)$ in Ω due to fluxes $\underline{j}(\underline{x},t)$ across $\partial\Omega$ and production/consumption of by $\hat{f}_s$ Ω.

Units of basic quantities:

- u is a density $\left[\frac{\#}{L^3}\right]$
- $\underline{j}$ is a flux $\left[\frac{\#}{L^2 T}\right]$
- $\hat{f}_s$ is a vol. rate $\left[\frac{\#}{L^3\ T}\right]$

General $\underline{\text{Macroscopic}}$ balance:

$$\frac{d}{dt} U = J + F$$

1) U is amount of $u(\underline{x},t)$ in Ω: $\underset{\#}{U(t)} = \int_\Omega \underset{\frac{\#}{L^3}}{\underline{u}} \underset{L^3}{dV}$

2) J is $\underline{\text{rate}}$ of transport of $\underline{u}$ across $\partial\Omega$.

$$\underset{\frac{\#}{T}}{J(t)} = -\oint_{\partial\Omega} \underset{\frac{\#}{L^2 T}}{\underline{j}} \cdot \underset{L^2}{\hat{n}\, dA}$$

3) F is $\underline{\text{rate}}$ of prod./cons. of $\underline{u}$ in Ω:

$$\underset{\frac{\#}{T}}{F(t)} = \int_\Omega \underset{\frac{\#}{L^3 T}}{\hat{f}_s} \underset{L^3}{dV}$$

$$\underset{\frac{\#}{T}}{\frac{dU}{dt}} = \underset{\frac{\#}{T}}{J} + \underset{\frac{\#}{T}}{F} \quad \checkmark$$

To obtain microscopic balance law

$\Rightarrow$ substitute def. of U, J, F

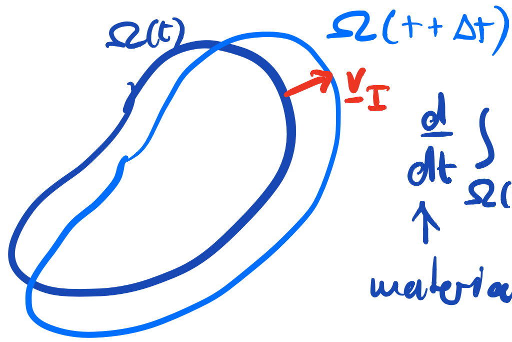
$$\boxed{\frac{d}{dt}\int_{\Omega} \underline{u}\, dV = -\oint_{\partial\Omega} \underline{j}\cdot\hat{n}\, dA + \int_{\Omega} \hat{f}_s\, dV}$$

Integral balance law

To obtain a local PDE we need to:

1) Exchange derivative integral
2 Transform surface integral to vol. int.

1) Reynolds transport theorem



If Ω is moving $\underline{v}_I(\underline{x})$

$$\underbrace{\frac{d}{dt}}_{\text{material}}\int_{\Omega(t)} \underline{u}(\underline{x},t)\, dV = \int_{\Omega} \underbrace{\frac{\partial u}{\partial t}}_{\text{partial}}\, dV + \oint_{\partial\Omega} \underline{u}(\underline{v}_I\cdot\underline{\hat{n}})\, dA$$

We consider the Eulerian limit $\underline{v}_I = 0$

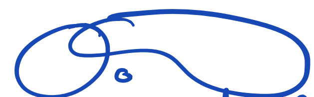
$$\rightarrow \frac{d}{dt}\int_{\Omega} u\, dV = \int \frac{du}{dt}\, dV \quad \checkmark$$

2) Divergence thm: $\oint_{\partial\Omega} \underline{j}\cdot n \, dA = \int_{\Omega} \nabla\cdot\underline{j} \, dV$

substitute: $\int_{\Omega} \underbrace{\left(\frac{\partial u}{\partial t} + \nabla\cdot\underline{j} - \hat{f}_s \right)}_{g(\underline{x},t)} dV = 0$

Localization: This holds for any Ω

$$\int_{\Omega} g(\underline{x},t)\, dV = 0$$



the integrand g must be zero everywhere.

Local form of general balance law

$$\boxed{\underbrace{\frac{\partial u}{\partial t}}_{\frac{\#}{L^3T}} + \underbrace{\nabla}_{\frac{1}{L}}\cdot\underbrace{\underline{j}(\underline{u})}_{\frac{\#}{L^2T}} = \underbrace{\hat{f}_s}_{\frac{\#}{L^3T}}}$$

u = unknown $[\#/L^3]$

$\underline{j}$ = flux $[\#/(L^2T)]$

$\hat{f}_s$ = source/sink $[\#/(L^3T)]$

Fluid Mass Balance in a saturated p.m.

Start from general balance and define

u, $\underline{j}$ and $\vec{f}_s$

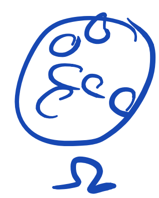
1) unknown to be balanced

$\underline{u} \equiv \phi\rho$ $\qquad$ ρ = mass density $\left[\frac{M}{L^3}\right]$

ϕ = porosity $[1]$

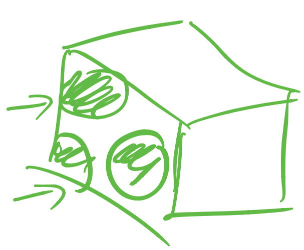
$\Rightarrow$ u is the fluid mass per unit volume of porous medium

2) Define the mass flux of fluid



$$\underline{j}(u) = \underline{j}(\phi\rho) = \rho\phi\,\underline{v} = \rho\,\underline{q}$$

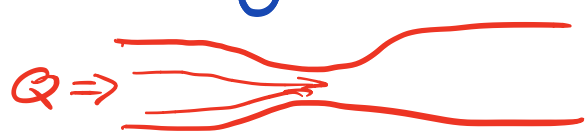
$\underline{v}$ = ave. interstitial fluid velocity $\left[\frac{L}{T}\right]$



$$\boxed{\underline{q} = \phi\,\underline{v}}$$

$\frac{L}{T} \quad 1 \quad \frac{L}{T}$

vol. flux given by Darcys law



3) Source term: $\hat{f}_s = \rho f_s$

$$\frac{M}{L^3 T} \quad \frac{M}{L^3} \frac{L^3}{L^3 T}$$

$f_s \left[\frac{1}{T}\right]$

Substitute into ~~the~~ general balance

$$\boxed{\frac{\partial}{\partial t}(\phi \rho) + \nabla \cdot (\rho q) = \rho f_s}$$

Fluid mass balance in sat. p.m.

To solve single PDE for single unknown we need constitutive laws.

1) Darcy's law: $q = -\underline{\underline{K}} \nabla h$

2) Equation of state: $\rho = \rho(p) = \rho(h)$

$h \sim p$ $\quad h = \frac{p - p_0}{\rho g}$

$$\frac{\partial}{\partial t}\left(\phi(\underline{x})\, \rho(h)\right) - \nabla \cdot \left(\rho(h)\, \underline{\underline{K}} \nabla h\right) = \rho(h) f_s$$

$\Rightarrow$ can be solved for $h(\underline{x}, t)$

Incompressible flow

- pressure variations during GW flow small $\Rightarrow$ $\rho = \rho_0 = \text{const.}$
- porosity & hydr. cond. are functions of space but not time

$$\cancelto{0}{\frac{\partial}{\partial t}\left(\phi(\underline{x})\,\cancel{\rho_0}\right)} + \nabla \cdot (\cancel{\rho_0}\, \underline{q}) = \cancel{\rho_0}\, f_s$$

$\Rightarrow$ continuity equation

$$\boxed{\nabla \cdot \underline{q} = f_s}$$

Substituting Darcy's law $\underline{q} = -K \nabla h$

$$\Rightarrow \quad \boxed{-\nabla \cdot (\underline{\underline{K}} \nabla h) = f_s}$$

Incompressible saturated GW flow (Poisson Eqn)

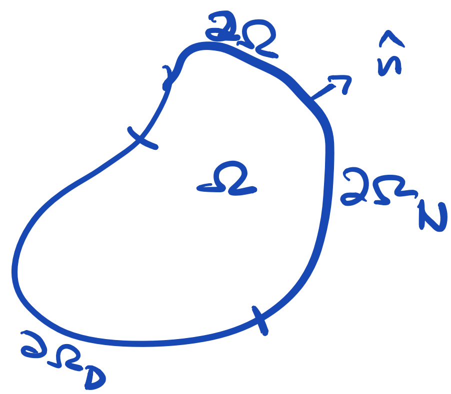
no time dependence $\rightarrow$ steady

Boundary value problem (BVP)

A well posed problem requires boundary cond.

PDE: $\nabla \cdot q = f_s$

$q = -K\nabla h$

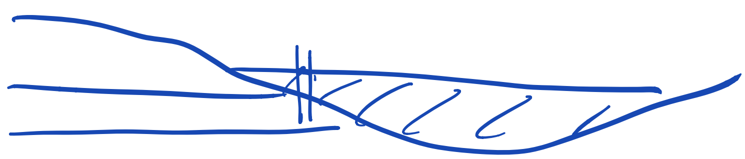


BC: a) Dirichlet BC

prescribe value of unknown on the boundary

$$h(\underline{x}) = h_B(\underline{x}) \qquad \underline{x} \in \partial\Omega_D$$

Example: Lake provides const head



b) Neuman BC

prescribes the value of flux on the boundary $\quad q \cdot \hat{n} = -q_B \quad \underline{x} \in \partial\Omega_N$

$q_B > 0$ is an inflow

Example: Rain fall