Incompressible Flow

- For the pressure variations encountered during groundwater flow the density of water is nearly constant. $\Rightarrow \quad \rho(p) \approx \rho_0 = \text{const.}$
- The porosity is highly variable in space but constant in time, in absence of reactions & compaction

$$\Rightarrow \quad \phi = \phi(\underline{x})$$

$\Rightarrow$ simplification of fluid mass balance

$$\frac{\partial}{\partial t}\cancelto{0}{(\phi \rho_0)} + \nabla \cdot (\rho_0 \vec{q}) = \rho_0 f \qquad \Rightarrow \qquad \boxed{\nabla \cdot \vec{q} = f}$$

- Darcy's law for constant density:

$h = z + \frac{p - p_0}{\rho_0 g}$ "hydraulic head" $[L]$

$K = \rho_0 g \frac{k}{\mu}$ "hydraulic conductivity" $[\frac{L}{T}]$ $\quad \boxed{\bar{q} = -K \nabla h}$

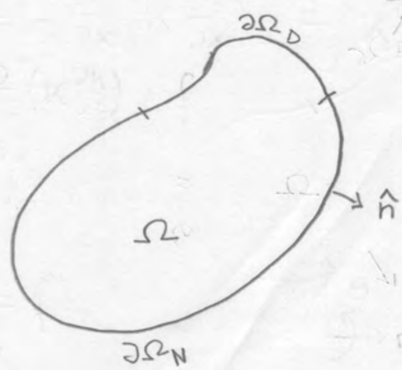
Equation for incompressible flow $\quad \boxed{-\nabla \cdot (K \nabla h) = f}$ Poisson Eqn.

Boundary Value Problem (BVP)

A well posed problem requires boundary conditions

PDE $\quad \nabla \cdot \bar{q} = f$

$\quad \bar{q} = -K \nabla h$



BC: a) Dirichlet (prescribed head)

$\quad h = h_B(\underline{x}) \quad \underline{x} \in \partial\Omega_D$

b) Neuman (prescribed flux)

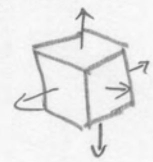
$\quad \bar{q} \cdot \hat{n} = -q_B \quad \underline{x} \in \partial\Omega_D$

Note: $q_B > 0$ corresponds to an inflow

Note on curvilinear coordinates

abstract

concrete



The derivation of balance laws in terms of div & grad is independent of coordinate system used? Benefit of abstract derivation over a derivation considering a concrete domain such as a box?

⇒ Equations written in div & grad are independent of the coordinate system

General $$-\nabla \cdot K \nabla h = f$$

Cartesian $$-\frac{\partial}{\partial x}\left(K \frac{\partial h}{\partial x}\right) - \frac{\partial}{\partial y}\left(K \frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(K \frac{\partial h}{\partial z}\right) = f$$

Cylindrical. $$-\frac{1}{r}\frac{\partial}{\partial r}\left(r K \frac{\partial h}{\partial r}\right) - \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(K \frac{\partial h}{\partial \theta}\right) - \frac{\partial}{\partial z}\left(K \frac{\partial h}{\partial z}\right) = f$$

Spherical : $$-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 K \frac{\partial h}{\partial r}\right) - \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \, K \frac{\partial h}{\partial \theta}\right) - \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \phi}\left(K \frac{\partial h}{\partial \phi}\right) = f$$

⇒ div-grad notation is compact and hides the complexities of the non-cartesian coordinate systems

⇒ we want the same for numerical implementation