Fluid & Solid Mass Balances

General balance eqn.: $\frac{\partial u}{\partial t} + \nabla \cdot \vec{j}(u) = \hat{f}_s(u)$

I. Fluid mass balance

1) Unknown to be balanced: $u = \phi \rho_f$

   "fluid mass per unit volume of porous medium"

2) Define mass flux of pore fluid

   $\vec{j}(u) = \vec{j}(\phi\rho_f) = \rho_f \phi \bar{v}_f = \rho_f \bar{q}_f$

3) Source term: $\hat{f}_s = \Gamma \quad \left[\frac{M}{L^3 T}\right]$ melt production rate

Fluid mass balance: $\boxed{\frac{\partial}{\partial t}(\phi\rho_f) + \nabla \cdot (\rho_f \phi \underline{v}_f) = \Gamma}$

II. Solid mass balance

1) unknown to be balanced: $u = (1-\phi)\rho_s$

2) mass flux of solid: $\vec{j}((1-\phi)\rho_s) = \rho_s (1-\phi) \vec{v}_s$

3) Source term: $\hat{f}_s = -\Gamma$

Solid mass balance: $\boxed{\frac{\partial}{\partial t}\left((1-\phi)\rho_s\right) + \nabla \cdot \left((1-\phi)\rho_s \bar{v}_s\right) = -\Gamma}$

If the phase densities are constant:

I) Fluid: $\frac{\partial \phi}{\partial t} + \nabla \cdot q_f = \frac{\Gamma}{\rho_f}$

II) Solid: $\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi v_s) = \frac{\Gamma}{\rho_s} + \nabla \cdot \underline{v}_s$

$\Rightarrow$ Two evolution equations for ϕ ?

Just need one - use the more convenient equation.

Two-phase continuity equation

Sum fluid & solid mass balance equations
$\Rightarrow$ steady-state equation

$$\nabla \cdot \left(\phi v_f + (1-\phi) v_s \right) = \frac{\Gamma}{\rho_f} - \frac{\Gamma}{\rho_s} = \frac{(\rho_s - \rho_f)\,\Gamma}{\rho_f \rho_s} = -\frac{\Delta\rho}{\rho_f \rho_s}\,\Gamma$$

Note on density difference:

$\Delta\rho = \rho_f - \rho_s > 0$ because in brine ice system $\rho_f > \rho_s$.

In our simple 2-phase system $\Delta\rho = \text{const.}$

Once salt is included the solid (ice + salt) can be lighter (ice-rich) or denser (salt-rich) than the fluid (brine)

Simplify continuity by introducing relative fluid flux:

$$\nabla \cdot (\phi \underline{v}_f - \phi \underline{v}_s + \underline{v}_s) = \nabla \cdot (\phi(\underline{v}_f - \underline{v}_s) + \underline{v}_s) = \nabla \cdot (q_r + \underline{v}_s)$$

so that:

$$\boxed{\nabla \cdot (q_r + \underline{v}_s) = -\frac{\Delta\rho}{\rho_f \rho_s}\,\Gamma}$$

We'll see the use of this equation below

Note: • q_r can be eliminated using Darcy's law
• Need an additional constitutive law for the volumetric strain rate $\dot{\varepsilon} \equiv \nabla \cdot \underline{v}_s$!