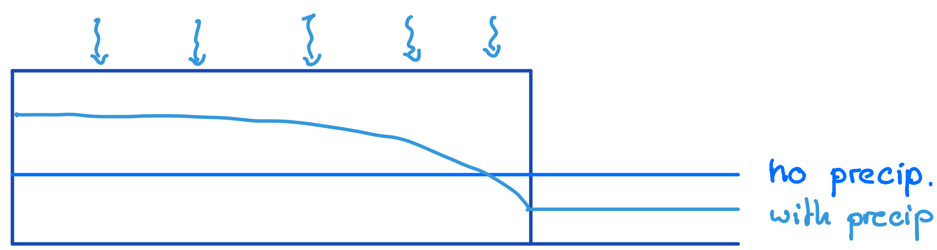
Groundwater - Surface water interaction

Consider the relation between Mars putative ocean and its global groundwater table. If the total water volume, V, on Mars is fixed then changes in ground-water volume V_G must affect ocean volume V_o.

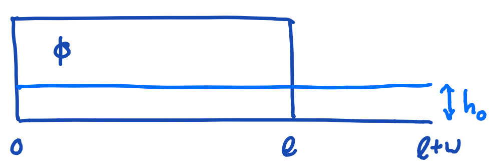
$$V = V_G + V_o = \text{const.}$$

This implies that the ocean level is coupled to the elevation of the ground water table. $\Rightarrow$ Non-linear Dirichlet BC!



Consider the following steady, linear, unconfined problem:

PDE: $-\nabla\cdot[K h \nabla h] = q_p$ on $x \in [0, \ell]$



BC: $q\cdot\hat{n}_i\big|_0 = 0 \qquad h(\ell) = h_o$

C: $V = V_G + V_o \qquad V_G = \int_0^\ell \phi\, h(x)\, dx \qquad V_o = h_o\, w$

Eliminate constraint: $h_o = \frac{V_o}{w} = \frac{V - V_G}{w} = \frac{V}{w} - \frac{\phi}{w}\int_0^\ell h(x)\, dx$

Final dimensional problem:

$$\boxed{\begin{array}{ll} \text{PDE:} & -\nabla\cdot[K h \nabla h] = q_p \qquad x \in [0, \ell] \\ \text{BC's:} & q\cdot\hat{n}_i\big|_0 = 0 \qquad h(\ell) = \frac{V}{\omega} - \frac{\phi}{\omega}\int_0^\ell h(x)\,dx \end{array}}$$

Introduce the dimensionless variables $h' = \frac{h}{h_c}$ $\quad x' = \frac{x}{\ell}$ $\quad q' = \frac{q}{q_c}$

$$q_c q = -\frac{K h_c}{\ell}\nabla' h' \qquad q_c = \frac{K h_c}{\ell}$$

PDE: $$-\frac{K h_c^2}{\ell^2}\nabla'\cdot[h'\nabla' h'] = q_p \qquad x' \in [0, \ell]$$

$$-\nabla'\cdot[h'\nabla' h'] = \frac{q_p \ell^2}{K h_c^2} \qquad dx = d(\ell x') = \ell\, dx'$$

BC: $$q'\cdot\hat{n}\big|_0 = 0 \qquad h_c h'(1) = \frac{V}{\omega} - \frac{\phi}{\omega} h_c \ell \int_0^1 h'\,dx'$$

$$h'(1) = \frac{V}{\omega h_c} - \frac{\phi\ell}{\omega}\int_0^1 h'\,dx$$

Two internal scales for h: 1) $\frac{q_p \ell^2}{K h_c^2} = 1 \Rightarrow h_c = \sqrt{\frac{q_p \ell^2}{K}}$

2) $\frac{V}{\omega h_c} = 1 \Rightarrow h_c = \frac{V}{\omega}$

Third scale: water height in absence of precipitation

$$\Rightarrow h(x) = h_0: \quad V = \phi h_0 \ell + \omega h_0 = (\phi\ell + \omega) h_0$$

$$h_0 = \frac{V}{\phi\ell + \omega} \equiv h_c$$

Substitute into PDE:

$$-\nabla'\cdot[h'\nabla' h'] = Pr$$

where: $\boxed{Pr = \frac{h_p}{h_0} = \frac{q_p \ell^2(\phi L + \omega)}{kV} \geq 0}$

Interpretation: change in h due to precipitation relative to "mean water level"

Substitute into BC:

$$h'(1) = \frac{\cancel{V}}{\omega \cancel{V}}(\phi \ell + \omega) - \frac{\phi L}{w}\int_0^1 h' dx' = 1 + \frac{\phi L}{\omega} - \frac{\phi L}{\omega}\int_0^1 h'\, dx'$$

$$h'(1) \equiv \Pi_0 = 1 + \frac{1}{Ca}\left(1 - \int_0^1 h' dx'\right)$$

$\boxed{Ca = \frac{\omega}{\phi L}}$ "Capacity number"

water in ocean to groundwater in absence of precipitation

Here Π_0 is the unknown water level in ocean that must be determined by mass balance!

Dimensionless problem (dropping primes)

$$\boxed{\begin{array}{ll} \text{PDE:} & -\nabla\cdot[h\nabla h] = Pr \qquad x \in [0,1] \\ \text{BC's:} & q\cdot\hat{n}|_0 = 0 \qquad h(1) = \Pi_b = 1 + \frac{1}{Ca}\left(1 - \int_0^1 h' dx'\right) \end{array}}$$

First solve for shape of GW table $h(x, \Pi_0)$, then determine Π_0 from mass balance.

Integrate: $$-h\frac{dh}{dx} = Pr\,x + c_1$$

Neu. BC: $$-h\,q = 0 + c_1 = 0 \quad \Rightarrow \quad c_1 = 0$$

Integrate: $$-h\,dh = Pr\,x\,dx$$

$$-\frac{h^2}{2} = Pr\frac{x^2}{2} + c_2$$

Dir. BC: $$-\frac{\Pi_0^2}{2} = Pr\frac{1}{2} + c_2 \quad \Rightarrow \quad c_2 = -\frac{1}{2}\left(Pr + \Pi_0^2\right)$$

$$-\frac{h^2}{\cancel{2}} = Pr\frac{x^2}{\cancel{2}} - \frac{1}{\cancel{2}}\left(Pr + \Pi_0^2\right)$$

$$h^2 = \Pi_0^2 + Pr\left(1 - x^2\right)$$

$$\Rightarrow \qquad \boxed{h = \sqrt{\Pi_0^2 + Pr\left(1 - x^2\right)}}$$

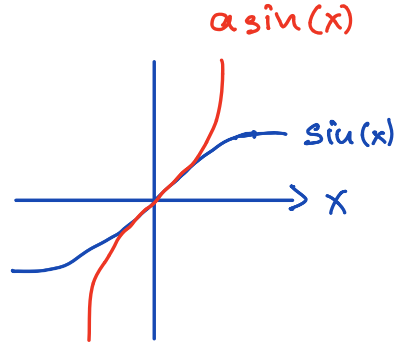
Now we need to determine Π_0 from mass balance. For this we require the integral

$$H = \int_0^1 h\,dx = \int_0^1 \sqrt{\Pi_0^2 + Pr\left(1 - x^2\right)}\,dx$$

$$\boxed{H(\Pi_0, Pr) = \frac{\sqrt{Pr}\left(Pr + \Pi_0^2\right)\operatorname{asin}\sqrt{\frac{Pr}{Pr + \Pi_0^2}} + \Pi_0 Pr}{2\,Pr}}$$

Limit $Pr \to 0$:

$$\lim_{Pr\to 0} H = \frac{\sqrt{Pr}}{2}\left(1+\frac{\Pi_0^2}{Pr}\right)\operatorname{asin}\sqrt{\frac{Pr}{Pr+\Pi_0}} + \frac{\Pi_0}{2}$$



for $x \ll 1$: $\operatorname{asin} x \sim x$

$$\lim_{Pr\to 0} H = \frac{\sqrt{Pr}}{2}\left(1+\frac{\Pi_0^2}{Pr}\right)\sqrt{\frac{Pr}{\cancel{Pr}+\Pi_0^2}} + \frac{\Pi_0}{2}$$

$$= \frac{1}{2}\left(\cancel{Pr}+\Pi_0\right) + \frac{\Pi_0}{2} = \Pi_0 \quad \checkmark$$

Numerical integration confirms this integral

Substitute $H(\Pi_0, Pr)$ into mass balance:

$$h(1) = \Pi_b = 1 + \frac{1}{Ca}\left(1 - \int_0^1 h' dx'\right)$$