Example 1: Linear confined aquifer

PDE : $-\frac{d}{dx}\left[bk\frac{dh}{dx}\right] = f_s \qquad x \in [0, \ell]$

BC: $\left.\frac{dh}{dx}\right|_0 = 0 \qquad h(L) = h_0$

dependent variable : h (head)

independent variable : x (distance)

Parameters : $b, k, \ell, f_s, h_0 \Rightarrow \underline{5}$

To determine the number of independent param.

we scale or non-dimensionalize the problem.

Use the parameters to render variable dimensionless.

Dimensionless variables:

$x' = \frac{x}{\ell} \quad \rightarrow \quad x = \ell\, x' \qquad \ell =$ "external scale"

$h' = \frac{h - h_0}{h_c} \quad \rightarrow \quad h = h_0 + h_c h' \qquad h_c =$ char. head

h_c is not yet clear → internal scale

substitute into PDE and BC's

PDE: $$-\frac{d}{d(\ell x')}\left[bK\,\frac{d(h_0+h_c h')}{d(\ell x')}\right]=f_s \qquad x'\ell\in[0,\ell]$$

since parameters are <u>constant</u> we can collect them

$$-\frac{bK}{\ell^2}\,\frac{d}{dx'}\left[\cancel{\frac{dh_0}{dx'}}^{\,0}+h_c\frac{dh'}{dx'}\right]=f_s$$

collect all params on rhs

$$-\frac{d^2h'}{dx'}=\frac{f_s\ell^2}{bKh_c}=1$$

l.h.s. is dimensionless ⇒ param. group on rhs also dim.less

setting $\frac{f_s\ell^2}{bKh_c}=1$ provides <u>internal</u> head scale $\boxed{h_c=\frac{f_s\ell^2}{bK}}$

BC: $$\left.\frac{dh}{dx}\right|_{x=0}=0$$

$$\left.\frac{d(h_0+h_c h')}{d(\ell x')}\right|_{\ell x'=0}=\frac{h_c}{\ell}\left.\frac{dh'}{dx'}\right|_{x'=0}=0 \Rightarrow \left.\frac{dh'}{dx'}\right|_0=0$$

$$h(x=\ell)=h_0$$

$$\cancel{h_0}+h_c h'(x'\ell=\ell)=\cancel{h_0}=0 \Rightarrow h'(x'=1)=0$$

Dimensionless problem:

$$\boxed{\begin{array}{lll}\text{PDE:} & -\dfrac{d^2h'}{dx'^2}=1 & x'\in[0,1]\\ \text{BC:} & \left.\dfrac{dh'}{dx'}\right|_0=0 & h'(1)=0\end{array}}$$

$$h=h_0+\frac{f_s\ell^2}{bK}h' \qquad x=\ell x'$$

scaling removes all parameters ⇒ single solution

Analytic solution: (dropping primes)

$$-\frac{d^2h}{dx^2} = 1$$

Integrate once: $-\frac{dh}{dx} = x + c_1$

use 1^{st} BC: $-\left.\frac{dh}{dx}\right|_{x=0} = c_1 = 0 \quad \Rightarrow \quad c_1 = 0$

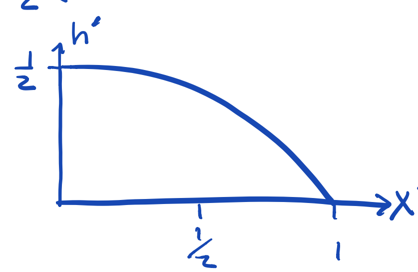
$$\Rightarrow \quad -\frac{dh}{dx} = x$$

Integrate again: $-h = \frac{x^2}{2} + c_2$

use 2^{nd} BC: $-h(1) = \frac{1}{2} + c_2 = 0 \quad \Rightarrow \quad c_2 = -\frac{1}{2}$

$$\Rightarrow \quad -h = \frac{x^2}{2} - \frac{1}{2} \qquad h = \frac{1}{2}(1-x^2)$$

dimensionless solution: $\boxed{h' = \frac{1}{2}(1-x'^2)}$



substitute to re-dimensonalize

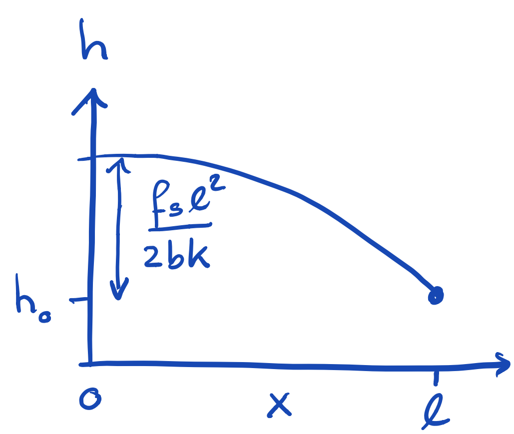
$$x' = \frac{x}{\ell} \qquad h' = \frac{h-h_0}{h_c} \qquad h_c = \frac{f_s \ell^2}{bK}$$

$$\frac{h-h_0}{h_c} = \frac{1}{2}\left(1-\left(\frac{x}{\ell}\right)^2\right)$$

$$h = h_0 + \frac{h_c}{2}\left(1-\left(\frac{x}{\ell}\right)^2\right)$$

Dimensional solution:

$$\boxed{h = h_0 + \frac{f_s \ell^2}{2bK}\left(1-\left(\frac{x}{\ell}\right)^2\right)}$$



Hence the internal head scale $\frac{f_s \ell^2}{bK}$ gives the order of magnitude for the increase in head across the aquifer.