Non-dimensionalization of ductile flow equations

Why do this? 1) Cleans up the equations

2) Identifies independent set of governing parameters → reduces parameters

3) Identifies terms that can be dropped

4) Better scaling of the equations ⇒ helps the numerical solution

Dimensional system in head formulation with constitutive laws $k = k_0 \phi^n$ and $\xi = \xi_0 / \phi^m$ substituted and simplified notation $\underline{v}_s = \underline{v}$ and $q_r = q$

$$\left.\begin{aligned} &1)\ \frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \underline{v}) = \frac{\Gamma}{\rho_s} + \frac{\phi^m}{\Xi_0}(h - z) \\ &2)\ -\nabla \cdot \left(K_0 \phi^n \nabla h \right) + \frac{\phi^m}{\Xi_0} h = -\frac{\Delta\rho}{\rho_s \rho_f} \Gamma + \frac{\phi^m}{\Xi_0} z \\ &3)\ -\nabla^2 u = \frac{\phi^m}{\Xi_0}(h - z) \end{aligned}\right\} \text{on } \Omega: \begin{aligned} &x \in [0, L] \\ &z \in [0, H] \\ &t \in [0, T] \end{aligned}$$

parameters $K_0 = \dfrac{k_0 \Delta\rho g}{\mu_f}$ $\quad \Xi_0 = \dfrac{\xi_0}{\Delta\rho g}$

Scale all variables:

independent variables: $\underline{x}_D = \dfrac{\underline{x}}{x_c}$ $\quad t_D = \dfrac{t}{t_c}$

primary dependent variables: $\phi_D = \dfrac{\phi}{\phi_c}$ $\quad h_D = \dfrac{h}{h_c}$ $\quad u_D = \dfrac{u}{u_D}$

secondary dependent variables. $\underline{v}_D = \dfrac{\underline{v}}{v_c}$ $\quad q_D = \dfrac{q}{q_c}$ $\quad \Gamma_D = \dfrac{\Gamma}{\Gamma_c}$

All variables with subscript D are dimensionless and optimally the characteristic quantities are chosen such that the magnitude of the dimensionless quantities is close to one.

What are these characteristic scales?

Some obvious external scales: $x_c \rightarrow H, L$
$t_c \rightarrow T$

But typically we choose internal scales suggested by the equations themselves. See below

Non-dimensionalize by substituting scaled variables

$\phi = \phi_c \phi_D \quad t = t_c t_D \quad x = x_c x_D \quad y = x_c y_D \quad z = x_c z_D$

so that $\frac{\partial \phi}{\partial t} = \frac{\partial(\phi_c \phi_D)}{\partial(t_c t_D)} = \frac{\phi_c}{t_c} \underbrace{\frac{\partial \phi_D}{\partial t_D}}_{\text{dimensionless}}$

$$\nabla \cdot = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{\partial}{\partial(x_c x_D)}, \frac{\partial}{\partial(x_c y_D)}, \frac{\partial}{\partial(x_c z_D)}\right)$$
$$= \frac{1}{x_c}\left(\frac{\partial}{\partial x_D}, \frac{\partial}{\partial y_D}, \frac{\partial}{\partial z_D}\right) = \frac{1}{x_c} \nabla_D \cdot$$

Start with overpressure equation

$$-\nabla \cdot (K_0 \phi^n \nabla h) + \frac{\phi^m}{\Xi_0} h = \frac{\phi^m}{\Xi_0} z$$

substitute

$$-\frac{K_0 \phi_c^n h_c}{x_c^2} \nabla_D \cdot (\phi_D^n \nabla_D h_D) + \frac{\phi_c^m h_c}{\Xi_0} \phi_D^m h_D = -\frac{\Delta\rho \Gamma_c}{\rho_f \rho_s} \Gamma_D + \frac{\phi_c^m x_c}{\Xi_0} \phi_D^m z_D$$

char. hydraulic conductivity $K_c = K_0 \phi_c^n$
char. hydraulic resistance: $\Xi_c = \frac{\Xi_0}{\phi_c^m}$

Set divergence term to unity by dividing by coefficient

$$-\nabla_D \cdot (\phi_D^n \nabla_D h_D) + \underbrace{\frac{x_c^2}{K_c \Xi_c}}_{\Pi} \phi_D^m h_D = -\underbrace{\frac{\Delta\rho \Gamma_c x_c^2}{\rho_f \rho_s K_c h_c}}_{\Pi_2} \Gamma_D + \underbrace{\frac{x_c^3}{K_c \Xi_c h_c}}_{\Pi_3} \phi_D^m z_D$$

Three dimensionless parameter groupings Π_1, Π_2 & Π_3

$\Pi_1 = \frac{x_c^2}{k_c \Xi_c}$ provides an internal scale for x_c

setting $\frac{x_c^2}{K_c \Xi_c} = 1 \Rightarrow x_c = \sqrt{K_c \Xi_c} = \sqrt{K_0 \phi_c^n \frac{\Xi_0}{\phi_c^m}}$ $\quad K_0 = \frac{k_0 \Delta\rho g}{\mu_f}$

$= \sqrt{\frac{k_0 \phi_c^n \cancel{\Delta\rho g}\, \xi_0}{\mu_f \cancel{\Delta\rho g}\, \phi_c^m}}$ $\quad \Xi_0 = \frac{\xi_0}{\Delta\rho g}$

compaction length: $\boxed{x_c = \sqrt{\frac{k_c \xi_c}{\mu_f}}}$ $\quad k_c = k_0 \phi_c^n \quad \xi_c = \xi_0/\phi^m$

The compaction length, $\delta = \sqrt{\frac{k_c \xi_c}{\mu_f}}$, is an internal length scale of the ductile flow system. Physical interpretation is the distance over which changes in porosity can be communicated in the ductile porous medium.

Hence we have:

$$-\nabla_D \cdot (\phi_D^n \nabla_D h_D) + \phi_D^m h_D = \underbrace{\frac{\Delta\rho \Gamma_c x_c^2}{\rho_f \rho_s K_c h_c}}_{\Pi_2} \Gamma_D + \underbrace{\frac{x_c}{h_c}}_{\Pi_3} \phi_D^m z_D$$

Once x_c is determined, Π_3 suggests $h_c = x_c = \delta$

We don't know much about Γ so we choose $\Pi_2 = 1$

$$\boxed{-\nabla_D \cdot (\phi_D^n \nabla_D h_D) + \phi_D^m h_D = -\Gamma_D + \phi_D^m z_D}$$ where $\Gamma_c = \frac{\rho_f \rho_s K_c}{\Delta\rho\, x_c}$

1) cleanedup the equation

2) reduced number of parameters to ?

3) See later if $h_D \sim 1$

Scale equation for velocity potential

$$-\nabla^2 u = \frac{\phi^m}{\Xi_0}(h-z)$$

$$-\frac{u_c}{x_c^2}\nabla_D^2 u_D = \underbrace{\frac{\phi_c^m}{\Xi_0}}_{\Xi_c}\phi_D^m(h_c h_D - x_c x_D) \quad \text{where } h_c - x_c$$

$$-\nabla_D^2 u_D = \frac{x_c^3}{u_c \Xi_c}\phi_D^m(h_D - z_D); \text{ suggests } \frac{x_c^3}{u_c\Xi_c} = 1 \Rightarrow u_c = \frac{x_c^3}{\Xi_c}$$

dimensionless equation

$$u_c = \frac{K_c \cancel{\Xi_c} x_c}{\cancel{\Xi_c}} = K_c x_c$$

$$\boxed{-\nabla_D^2 u_D = \phi_D^m(h_D - z_D)}$$

1) Cleaned up the equation
2) reduced number of parameters to ?
3) See later if $u_D \sim 1$

This also implies scale for the solid velocity

$$\underline{v} = \nabla u \qquad v_c \underline{v}_D = -\frac{u_c}{x_c}\nabla_c u_c \Rightarrow v_c = \frac{u_c}{x_c} = K_c$$

Scale porosity evolution equation

$$\frac{\partial\phi}{\partial t} + \nabla\cdot(\phi\underline{v}) = \frac{\Gamma}{\rho_s} + \frac{\phi^m}{\Xi_0}(h-z)$$

$$\frac{\phi_c}{t_c}\frac{\partial\phi_D}{\partial t_D} + \frac{\phi_c v_c}{x_c}\nabla_D\cdot(\phi_D\underline{v}_D) = \frac{\Gamma_c}{\rho_s}\Gamma_D + \frac{x_c}{\Xi_c}\phi_D^m(h_D - z_D)$$

set time derivative term to unity by dividing by coefficient

$$\frac{\partial\phi_D}{\partial t_D} + \underbrace{\frac{v_c t_c}{x_c}}_{\Pi_A}\nabla_D\cdot(\phi_D v_D) = \underbrace{\frac{\Gamma_c t_c}{\rho_s\phi_c}}_{\Pi_R}\Gamma_D + \underbrace{\frac{x_c t_c}{\phi_c\Xi_c}}_{\Pi_c}\phi_D^m(h_D - z_D)$$

Three dimensionless parameter groups that suggest internal time scales.

Possible time scales:

1) Advective time scale: $\Pi_A = \frac{v_c t_c}{x_c} = 1 \Rightarrow t_c = \frac{x_c}{v_c}$ "time for solid to flow one compaction length"

2) Reactive time scale: $\Pi_B = \frac{\Gamma_c t_c}{\rho_s \phi_c} = 1 \Rightarrow t_c = \frac{\rho_s \phi_c}{\Gamma_c}$ "time to create ϕ_c by melting"

3) Compaction time scale: $\Pi_C = \frac{x_c t_c}{\phi_c \Xi_c} = 1 \Rightarrow t_c = \frac{\phi_c \Xi_c}{x_c}$

If solid deformation is only induced by melt migration advection is small. and our initial models neglect melting, therefore we choose compaction time scale, $t_c = \frac{\phi_c \Xi_c}{x_c}$

$$\frac{\partial \phi_D}{\partial t} + \underbrace{\frac{v_c \phi_c \Xi_c}{x_c^2}}_{Pe} \nabla_D \cdot (\phi_D \underline{v}_D) = \underbrace{\frac{\Gamma_c \Xi_c}{\rho_s x_c}}_{Da} \Gamma_D + \phi_D^m (h_D - z_D)$$

Peclet number: $Pe = \frac{v_c \phi_c \Xi_c}{x_c^2} = \frac{K_c \Xi_c \phi_c}{x_c^2} = \phi_c \ll 1$

Damköhler number: $Da = \frac{\Gamma_c \Xi_c}{\rho_s x_c} = \frac{\rho_f \cancel{\rho_s} K_c \Xi_c}{\Delta\rho\, x_c \cancel{\rho_s} x_c} = \frac{\rho_f}{\Delta\rho} \overbrace{\cancel{\frac{K_c \Xi_c}{x_c^2}}}^{1} = \frac{\rho_f}{\Delta\rho}$

So we have the dimensionless ϕ evolution equation:

$$\frac{\partial \phi_D}{\partial t_D} + Pe\, \nabla \cdot (\phi_D \underline{v}_D) = Da\, \Gamma_D + \phi_D^m (h_D - z_D)$$

Hence we have the following dimensionless governing equations

$$\left.\begin{aligned} &1)\ \frac{\partial \phi_D}{\partial t_D} + Pe\, \nabla \cdot (\phi_D \underline{v}_D) = Da\, \Gamma_D + \phi_D^m (h_D - z_D) \\ &2)\ -\nabla_D \cdot (\phi_D^n \nabla_D h_D) + \phi_D^m h_D = -\Gamma_D + \phi^m z_D \\ &3)\ -\nabla_D^2 u_D = \phi_D^m (h_D - z_D) \end{aligned}\right\} \text{ on } \Omega \quad \begin{aligned} x_D &\in [0, \tfrac{L}{\delta}] \\ z_D &\in [0, \tfrac{H}{\delta}] \\ t_D &\in [0, \tfrac{T}{t_c}] \end{aligned}$$

where $\underline{v}_D = -\nabla u_D$

$\Rightarrow$ three dimensionless governing parameters: 1) Pe
2) Da
3) $\frac{H}{\delta} \left(\frac{L}{\delta}\right)$