Summary: Solving Linear System with constraints

PDE $\underline{\underline{L}}\,\underline{h} = \underline{f}$

BCs: $\underline{\underline{B}}\,\underline{h} = \underline{g}$

Step 1 Find particular solution

$$\underline{h}_p = \underline{\underline{B}}^T \underline{h}_{pr} \quad \text{and} \quad \underline{h}_{pr} = (\underline{\underline{B}}\,\underline{\underline{B}}^T)^{-1} \underline{g}$$

$$\Rightarrow \boxed{\underline{h}_p = \underline{\underline{B}}^T (\underline{\underline{B}}\,\underline{\underline{B}}^T)^{-1} \underline{g}}$$

Step 2: Find associated homogeneous solution

$$\underline{\underline{N}}^T \underline{\underline{L}}\,\underline{\underline{N}}\ \underline{\underline{N}}^T \underline{h}_0 = \underline{\underline{N}}^T (\underline{f}_s + \underline{f}_D) \qquad \text{where } \underline{f}_D = -\underline{\underline{L}}\,\underline{h}_p$$

$$\boxed{\underline{h}_0 \quad \underline{\underline{N}}\left[(\underline{\underline{N}}^T \underline{\underline{L}}\,\underline{\underline{N}})^{-1}\,\underline{\underline{N}}^T (\underline{f}_s - \underline{\underline{L}}\,\underline{h}_p)\right]}$$

Step 3: Add homogeneous & particular solutions

$$\boxed{\underline{h} = \underline{h}_0 + \underline{h}_p}$$

All this will be encapsulated in a general purpose Matlab function to solve linear boundary value problems

h = solve_lbvp(L, f, B, g, N)

Note on source terms:

The discrete rhs $\underline{f}_s$ is the cell average of $f(\underline{x})$!