Spherical shell solution

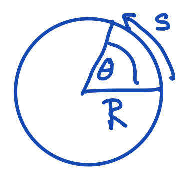
Steady confined aquifer with precipitation on a spherical shell.

PDE: $-\frac{1}{R\sin\theta}\frac{d}{d\theta}\left[\sin\theta\; bK\,\frac{1}{R}\frac{dh}{d\theta}\right] = q_p$ on $\theta \in [\theta, \theta_b]$

BC: $\left.\frac{dh}{d\theta}\right|_0 = 0$ $\qquad h(\theta_b) = h_0$

Dimensionless equations:

Note the definitions of radians $\theta = \frac{s}{R}$



hence θ is dimensionless and order one.

$\Rightarrow$ only need to scale h: $\qquad h' = \frac{h - h_0}{h_c}$

substitute:

$$-\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\,\frac{dh}{d\theta}\right] = \frac{q_b R^2}{K b h_c} = 1 \qquad \Rightarrow \boxed{h_c = \frac{q_b R^2}{K b}}$$

$$\boxed{\begin{array}{ll} \text{PDE}: & -\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\,\frac{dh'}{d\theta}\right] = 1 \qquad \theta \in [0, \theta_b] \\ \text{BC}: & q' = -\left.\frac{dh'}{d\theta}\right|_0 = 0 \qquad h'(\theta_b) = 0 \end{array}}$$

Integrate: $\quad -\dfrac{d}{d\theta}\left(\sin\theta \dfrac{dh'}{d\theta}\right) = \sin\theta$

$$+\sin\theta \frac{dh'}{d\theta} = +\cos\theta + c_1$$

Neumann BC: $\quad \sin 0 = 0 \qquad \cos 0 = 1$

$$0 \cdot 0 = 1 + c_1 \quad \Rightarrow \quad \boxed{c_1 = -1}$$

$$\Rightarrow \quad \sin\theta \frac{dh'}{d\theta} = \cos\theta - 1$$

$$\frac{dh'}{d\theta} = \frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta} = \cot\theta - \csc\theta$$

Integrate by parts: $\quad h' = \int \cot\theta - \csc\theta \, d\theta + c_2$

$$h' = \log(\cos\theta + 1) + c_2$$

Dirichlet BC at θ_b: $\quad 0 = \log(\cos\theta_b + 1) + c_2$

$$c_2 = -\log(\cos\theta_b + 1)$$

$$\boxed{\begin{aligned} h' &= \ln\left(\frac{\cos\theta + 1}{\cos\theta_b + 1}\right) \\ q' &= \csc\theta - \cot\theta \end{aligned}}$$