Steady unconfined flow on spherical shell

Consider a steady unconsined aquifer on a spherical shell with precipitation, so that we have

PDE: $-\frac{1}{R\sin\theta}\frac{d}{d\theta}\left[\sin\theta\,\frac{K}{R}\,h\,\frac{dh}{d\theta}\right] = q_p \qquad \theta \in [0, \theta_b]$

BC's: $\left.\frac{dh}{d\theta}\right|_0 = 0 \qquad h(\theta_b) = h_b$

Introduce dimensionless head $\quad h' = \frac{h}{h_c}$

$$-\frac{K\,h_c^2}{R^2\sin\theta}\frac{d}{d\theta}\left[\sin\theta\,h\,\frac{dh'}{d\theta}\right] = q_p \qquad \theta \in [0, \theta_b]$$

$$-\frac{d}{d\theta}\left[\sin\theta\,h\,\frac{dh}{d\theta}\right] = \frac{q_p R^2}{K\,h_c^2}\,\sin\theta \qquad \Rightarrow \qquad \boxed{h_c = \sqrt{\frac{q_p R^2}{K}}}$$

Dimensionless problem

PDE: $-\frac{d}{d\theta}\left[\sin\theta\,h\,\frac{dh}{d\theta}\right] = \sin\theta \qquad \theta \in [0, \theta_b]$

BC's: $\left.\frac{dh}{d\theta}\right|_0 = 0 \qquad h'(\theta_b) = \Pi \qquad \boxed{\Pi = \frac{h_b}{h_c} = \frac{h_b}{R}\sqrt{\frac{K}{q_p}}}$

Integrate: $-\sin\theta\,h\,\frac{dh}{d\theta} = -\cos\theta - c_1$

Neu. BC ($\theta=0$): $\cancelto{0}{\sin\theta}\,h\,\left.\cancelto{c}{\frac{dh}{d\theta}}\right|_0 = \cancelto{1}{\cos\theta} + c_1 \quad \Rightarrow \quad c_1 = -1$

$$\sin\theta\,h\,\frac{dh}{d\theta} = \cos\theta - 1$$

$$h\frac{dh}{d\theta} = \frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta} = \cot\theta - \csc\theta$$

Integrate by sep. of variables:

$$h\,dh = (\cot\theta - \csc\theta)\,d\theta$$

$$\frac{h^2}{2} = \log(\cos\theta + 1) + c_2$$

Dir. BC $(\theta = \theta_b)$: $\frac{\pi^2}{2} = \log(\cos\theta_b + 1) + c_2$

$$\Rightarrow c_2 = \frac{\pi^2}{2} - \log(\cos\theta_b + 1)$$

$$\frac{h^2}{2} = \frac{\pi^2}{2} + \log(\cos\theta + 1) - \log(\cos\theta_b + 1)$$

$$\Rightarrow \boxed{\begin{aligned} h' &= \sqrt{\pi^2 + 2\log\left(\frac{\cos\theta + 1}{\cos\theta_b + 1}\right)} \\ q' &= -\frac{dh}{d\theta} = \frac{1 - \cos\theta}{h(\theta)\sin\theta} \end{aligned}}$$