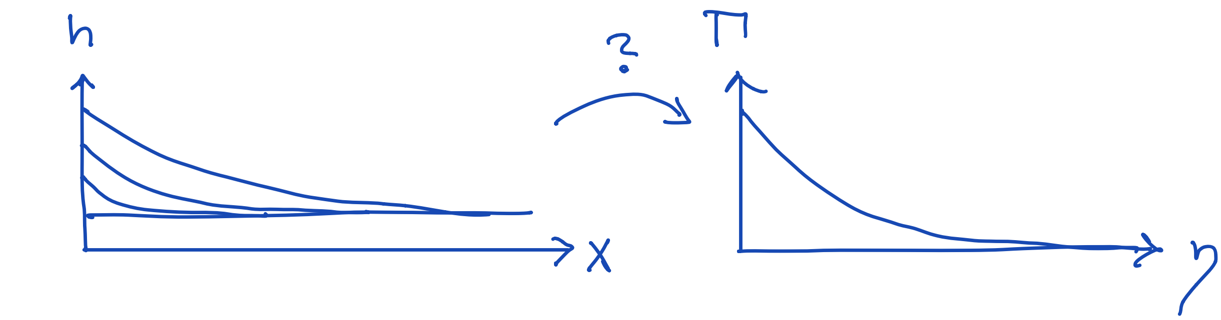
Transient recharge of linear confined aquifer

Consider a confined semi-infinite aquifer with constant head subjected to sudden recharge with constant rate.

PDE: $\frac{\partial h}{\partial t} - \nabla \cdot D_{hyd} \nabla h = 0 \qquad x > 0$

BC's: $q_i = -K \nabla h \cdot \hat{x}$

IC: $h(x,0) = h_0$



Can this PDE for $h(x,t)$ be reduced to an ODE for $\Pi(\eta)$?

As before $\eta = \frac{x}{\sqrt{4Dt}}$ but we also need a new variable for the head because $h(0,t)$ keeps changing.

Try $\Pi = \frac{h - h_0}{c\, t^{\alpha}}$ where we need to determine c and α from the BC.

$$h = h_0 + ct^\alpha \,\Pi(\eta(t,x))$$

$$\frac{\partial h}{\partial x} = \frac{\partial h}{\partial \Pi}\frac{d\Pi}{d\eta}\frac{\partial \eta}{\partial x} = \frac{1}{\sqrt{4Dt}}\, c\, t^\alpha \frac{d\Pi}{d\eta}$$

On the boundary: $q_i = -K \left.\frac{\partial h}{\partial x}\right|_0$ $\quad \left.\frac{\partial h}{\partial x}\right|_0 = -\frac{q_i}{K}$

$$\frac{ct^\alpha}{\sqrt{4Dt}} \left.\frac{d\Pi}{d\eta}\right|_{\eta=0} = -\frac{q_i}{K}$$

$\Rightarrow$ t's must cancel $\quad \alpha = \frac{1}{2}$

$$\frac{c}{\sqrt{4D}} \left.\frac{d\Pi}{dt}\right|_0 = -\frac{q_i}{K} \quad \longrightarrow \quad \left.\frac{d\Pi}{d\eta}\right|_0 = -\frac{q_i\sqrt{4D}}{cK} = 1$$

$$\Rightarrow \quad c = \frac{q_i\sqrt{4D}}{K} = \frac{2q_i\sqrt{K}}{K\sqrt{S_s}} = 2\frac{q_i}{\sqrt{KS_s}}$$

Self-similar variable:

$$\Pi = \frac{h-h_0}{ct^\alpha} = \frac{h-h_0}{\frac{2q_i}{\sqrt{KS_s}}\sqrt{t}} = \frac{(h-h_0)\sqrt{KS_s}}{2q_i\sqrt{t}}$$

Transform derivatives:

$$\frac{\partial \eta}{\partial x} = \frac{1}{\sqrt{4Dt}} \qquad \frac{\partial \eta}{\partial t} = -\frac{\eta}{2t}$$

$$h = h_0 + c\,\Pi\sqrt{t}$$

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial t}\left(h_0 + c\,\Pi\sqrt{t}\right) = c\,\frac{\partial}{\partial t}\left(\Pi(\eta(x,t))\sqrt{t}\right)$$

$$= c\left(\frac{1}{2}\frac{1}{\sqrt{t}}\Pi(\eta) + \sqrt{t}\,\frac{d\Pi}{d\eta}\frac{\partial\eta}{\partial t}\right) = c\left(\frac{1}{2}\frac{1}{\sqrt{t}}\Pi + \frac{\sqrt{t}\,\eta}{2t}\frac{d\Pi}{d\eta}\right)$$

$$= \frac{c}{2\sqrt{t}}\left(\Pi + \eta\frac{d\Pi}{d\eta}\right)$$

$$\frac{\partial^2 h}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left(h_0 + c\,\Pi(\eta(x,t))\sqrt{t}\right) = c\sqrt{t}\,\frac{\partial^2}{\partial x^2}\left(\Pi(\eta(x,t))\right)$$

$$= c\sqrt{t}\,\frac{d^2\Pi}{d\eta^2}\left(\frac{\partial\eta}{\partial x}\right)^2 = \frac{c\sqrt{t}}{4Dt}\frac{d^2\Pi}{d\eta^2} = \frac{c}{4D\sqrt{t}}\frac{d^2\Pi}{d\eta^2}$$

substitute into PDE:

$$\frac{\partial h}{\partial t} - D\frac{\partial^2 h}{\partial x^2} = 0$$

$$\frac{\cancel{c}}{2\cancel{\sqrt{t}}}\left(\Pi + \eta\frac{d\Pi}{d\eta}\right) - \frac{\cancel{c}\cancel{D}}{4\cancel{D}\cancel{\sqrt{t}}}\frac{d^2\Pi}{d\eta^2} = 0$$

Self-similar Problem

$$\boxed{\begin{aligned} &\text{ODE:} \quad \frac{d^2\Pi}{d\eta^2} - 2\eta\frac{d\Pi}{d\eta} - 2\Pi = 0 \\ &\text{BC's:} \quad \left.\frac{d\Pi}{d\eta}\right|_0 = -1 \qquad \lim_{\eta\to\infty}\Pi = 0 \end{aligned}}$$

Mathematica: $\Pi(\eta) = \frac{e^{-\eta^2}}{\sqrt{\pi}} - \eta \operatorname{erfc}(\eta)$

Resubstitute: $\eta = \frac{x}{\sqrt{4Dt}}$ $\quad \Pi = \frac{(h-h_0)\sqrt{KS_s}}{2q_i\sqrt{t}}$

$$\boxed{h = h_0 + \frac{2q_i\sqrt{t}}{\sqrt{KS_s}}\left(\frac{e^{-\frac{x^2}{4Dt}}}{\sqrt{\pi}} - \frac{x}{\sqrt{4Dt}}\operatorname{erfc}\left(\frac{x}{\sqrt{4Dt}}\right)\right)}$$