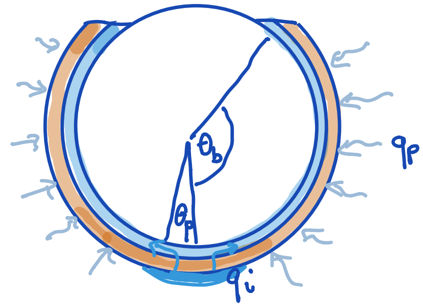
Transient spherical cap aquifer

Consider a confined aquifer with precipitation that adjusts to a sudden recharge due to basal polar melting.



Integrated over the depth of the aquifer we have:

PDE: $$b S_s \frac{\partial h}{\partial t} - \nabla \cdot [bK \nabla h] = q_p \qquad \theta \in [\theta_p, \theta_b]$$

BC's: $$q_i = -K \nabla h \big|_{\theta_p} \qquad h(\theta_b) = h_b$$

IC: $$h = h_b + \frac{q_p R^2}{bK}\left[\log\left(\frac{\sin\theta}{\sin\theta_p}\right) + \cos\theta_p \log\left(\frac{\csc\theta + \cot\theta}{\csc\theta_b + \cot\theta_b}\right)\right]$$

Note, the IC is the soln to steady problem with recharge.

Scaling: $h_c = \frac{q_p R^2}{bK} \qquad h' = \frac{h - h_b}{h_c}$

$\nabla' = R\nabla \quad \rightarrow \quad \nabla' = \frac{d}{d\theta}$

$\nabla' \cdot = R \nabla \cdot \quad \rightarrow \quad \nabla' \cdot = \frac{1}{\sin\theta} \frac{d}{d\theta}\left(\sin\theta \quad \right)$

$t' = t/t_c \qquad t_c$ undetermined for now substitute

$$\frac{b S_s h_c}{t_c} \frac{\partial h'}{\partial t'} - \frac{b K h_c}{R^2} \nabla' \cdot \nabla' h = q_p \qquad \theta \in [\theta_p, \theta_b]$$

scale to accumulation term

$$\frac{\partial h'}{\partial t'} - \frac{K t_c}{S_s R^2} \nabla' \cdot \nabla' h' = \frac{K t_c}{S_s R^2}$$

Both dimensionless groups are same and suggest diffusive time scale: $t_c = \frac{R^2}{D_{hyd}}$ $\qquad D_{hyd} = \frac{K}{S_s} \quad \left[\frac{L^2}{T}\right]$

↑ hydraulic diffusivity

BC: $q_i = -K \nabla h \big|_{\theta_p} \qquad q_i = -\frac{K h_c}{R} \nabla' h' \big|_{\theta_p}$

$$-\nabla' h' \big|_{\theta_p} = \frac{q_i R}{K h_c} = \frac{q_i R}{K} \frac{b K}{q_p R^2} = \frac{q_i b}{q_p R} = \Pi$$

Dimensionless transient problem

PDE: $\frac{\partial h'}{\partial t'} - \nabla'^2 h' = 1 \qquad \theta \in [\theta_p, \theta_b]$

BC: $-\nabla' h' \big|_{\theta_p} = \Pi \qquad h'(\theta_b) = 0$

IC: $h'(\theta, 0) = \log\left(\frac{\sin\theta}{\sin\theta_b}\right) + \cos\theta_p \log\left(\frac{\csc\theta + \cot\theta}{\csc\theta_b + \cot\theta_b}\right)$