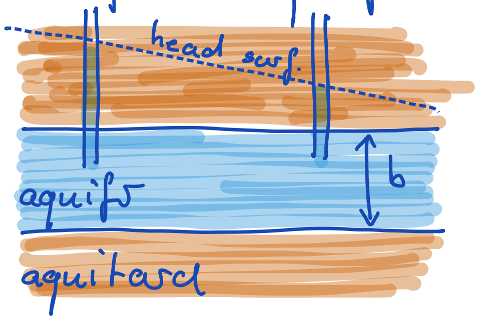
Flow in an unconfined aquifer

Confined aquifer

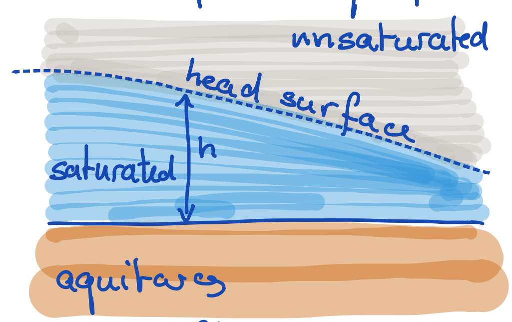


Steady flow:

$$-\nabla\cdot[b K \nabla h] = f_s$$

linear in h

Unconfined aquifer



Steady flow

$$-\nabla\cdot[h K \nabla h] = f_s$$

non-linear in h !

Example: Linear unconfined aquifer with precipitation

PDE: $-\frac{d}{dx}\left(K h \frac{dh}{dx}\right) = q_p \qquad x \in [0, \ell]$

BC: $\left.\frac{dh}{dx}\right|_0 = 0 \qquad h(\ell) = h_b$

Scale problem: $h' = \frac{h}{h_c} \qquad x' = \frac{x}{\ell}$

$$-\frac{h_c^2 K}{\ell^2} \frac{d}{dx'}\left[h' \frac{dh'}{dx'}\right] = q_p \qquad x' \in [0, 1]$$

$$-\frac{d}{dx}\left[h' \frac{dh'}{dx'}\right] = \frac{q_p \ell^2}{K h_c^2} = 1 \quad \Rightarrow \quad \boxed{h_c = \sqrt{q_p \ell^2 / K}}$$

Dimensionless problem:

PDE: $-\frac{d}{dx'}\left[h'\frac{dh'}{dx'}\right] = 1 \qquad x' \in [0,1]$

BC: $\frac{dh'}{dx'} = 0 \qquad h'(1) = \frac{h_b}{h_c} = \boxed{\frac{h_b}{\ell}\sqrt{\frac{K}{q_p}} = \Pi}$

$\Rightarrow$ This problem has a dimensionless parameter Π,

<u>unlike</u> the equivalent confined problem.

Note: Here $h_b \geq 0$ is the elevation above the base of the aquifer!

Note we have a non-linear ODE but it can be solved.

(dropping primes)

Integrate: $-h\frac{dh}{dx} = x + c_1$

Neu. BC $x=0$: $-h(0)\,0 = 0 + c_1 \quad \Rightarrow \quad c_1 = 0$

Integrate: $-h\,dh = x\,dx$

$$-\frac{h^2}{2} = \frac{x^2}{2} + c_2$$

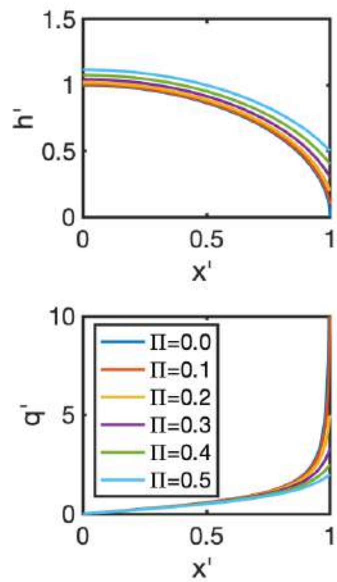
Dir. BC at $x=1$: $-\frac{\Pi^2}{2} = \frac{1}{2} + c_2 \quad \Rightarrow \quad c_2 = -\frac{1}{2}(\Pi^2+1)$

$$h^2 = \Pi^2 + 1 - x^2 \quad \Rightarrow \quad h = \sqrt{\Pi^2+1-x^2}$$

Flux: $q = -\frac{dh}{dx} = \frac{x}{h}$

Dimensionless solution

$$\boxed{\begin{aligned} h' &= \sqrt{1 + \Pi^2 - x'^2} \\ q' &= \frac{x'}{\sqrt{1+\Pi^2 - x'^2}} \end{aligned}}$$



where $\Pi = \frac{h_b}{\ell}\sqrt{\frac{K}{q_p}}$

Dimensional solution

resubstituting $h' = \frac{h}{\sqrt{\frac{q_p \ell^2}{K}}}$ $\quad x' = \frac{x}{\ell}$

$$h = \sqrt{\frac{q_p \ell^2}{K}} \sqrt{\frac{h_b}{\ell}\sqrt{\frac{K}{q_p}} + 1 - \left(\frac{x}{\ell}\right)^2} \qquad q = \frac{x}{\ell\, h}$$