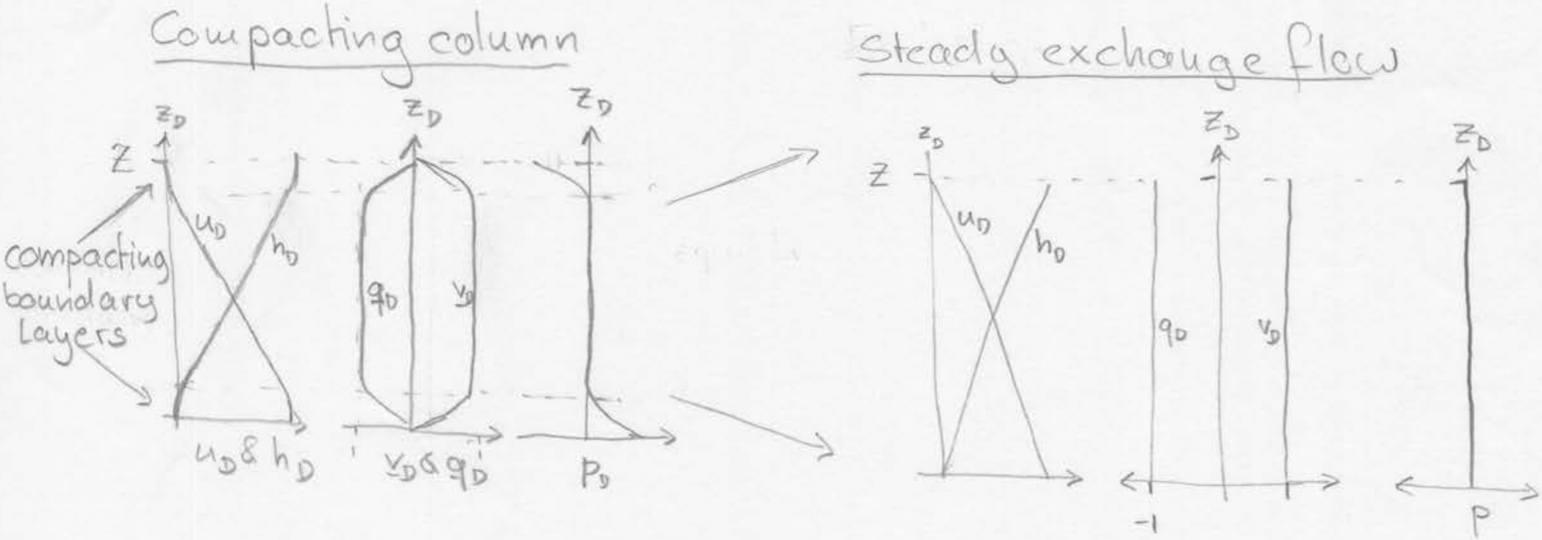
Neumann Boundary Conditions

- Dirichlet BC's prescribe the value of the unknown on the boundary ⇒ eliminate the unknowns along boundary.
- Neumann BC's prescribe the derivative of the unknown, so the unknown on boundary still needs to be determined.
- In our problems the derivative of h_D and u_D correspond two fluxes. Neumann BC's describe fluxes into/out of the domain

Consider the example problem of an open column with a steady exchange flow. This equivalent to the interior of the compacting column. (HW3)



Simple analytic solution. $p_D = 0$ $q_D = -1$ $v_D = 1$ $\phi_D = 1$

$$h_D = z_D \qquad u_D = Z - z_D$$

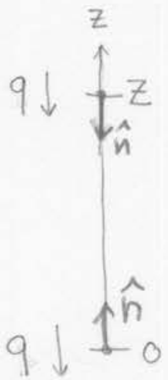
Solve mod Helmholtz eqn with Neumann BC's

PDE: $-\nabla_D \cdot \nabla_D h_D + h_D = 0 \qquad z_D \in [0, Z]$

BC's: $(q_D \cdot \hat{n}|_{z=0,Z} = q_b \Rightarrow (-\nabla_D h_D \cdot \hat{n}|_{0,Z} = q_b)$

Note: We assume inflows are positive ($q_b > 0$) and outflows are negative ($q_b < 0$), because it is physically intuitive.

This means $\hat{n}$ is the <u>inward</u> normal!



In our example brine flows downward ($q<0$) but enters the domain at the top ($q_b(Z) > 0$) and leaves it at base ($q_b(0) < 0$)

at $z = Z$: $\underset{\sim}{q} \cdot \hat{n}|_Z = q_{b,Z} > 0$ $\quad$ q and $\hat{n}$ point same direction

at $z = 0$: $\underset{\sim}{q} \cdot \hat{n}|_0 = q_{b,0} < 0$ $\quad$ q and $\hat{n}$ point opposite direction

in our case $|q_b| = 1$.

Param dof-neu = [dof-xmin, dof-xmax]

Param qb = [-1, 1]

We impose this BC as a source/sink term, f_n, in the boundary cell.

Total flow rate aross boundary face: $Q = A\, q_b$ $\quad$ A = area of face

Total fluid production in boundary cell: $F_n = V f_n$ $\quad$ V = volume of cell

They are eqivalent if $Q = F_n$ so that $\boxed{f_n = q_b \frac{V}{A}}$

The sign is automatically correct because $\hat{n}$ is inward normal so that inflow ($q_b > 0$) leads to a pos source $f_n > 0$ and an outflow ($q_b < 0$) leads to a neg sink $f_n < 0$.

This needs to be added to build_bnd. Given the vectors in Param that identify the cells and faces on Neumann bnd we can set all flux boundary conditions in one line.

Nn = number of Neuman bc's

Param dof_neu = Nn by 1 vector of cells

Param.dof_f_neu = Nn by 1 vector of faces

Param.qb - Nn by 1 vector of boundary fluxes

fn = N by 1 r.h.s. vector with Nn on zero entries

We construct fn as follows

$$\boxed{fn(dof\_neu) = qb * A(dof\_f\_neu)/V(dof\_neu)}$$

Here we assume V and A are new vectors that have been added to Grid.