Discretization of advective flux

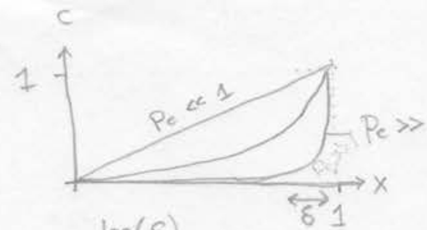
Example problem: Submarine groundwater discharge

Fresh water discharges at constant flow rate into a salty ocean

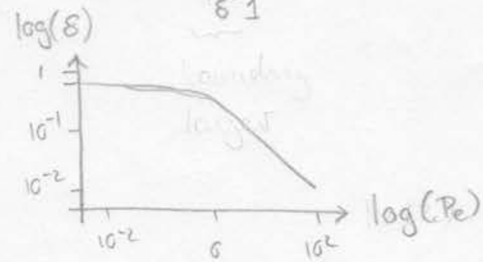
PDE. $\frac{d}{dx}\left(Pe\, c - \frac{dc}{dx}\right) = 0, \quad 0 \le x \le 1$

BC. $c(0) = 0 \qquad c(1) = 1$

Analytic solution. $\boxed{c(x) = \frac{e^{Pe\,x} - 1}{e^{Pe} - 1}}$



Diffusive boundary layer $\delta = -\frac{1}{Pe} \ln\left(1 + \varepsilon\left(e^{Pe} - 1\right)\right)$



Discrete advective fluxes

PDE: $\nabla \cdot (qc + D_* \nabla c) = 0$ where q's are known from flow problem

Discrete: $\underline{\underline{D}}(\underline{\underline{A}}(q)\, \underline{c} - D_* \underline{\underline{G}}\, c) = 0$

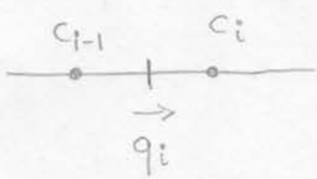
$$\underbrace{\left[\underline{\underline{D}}(\underline{\underline{A}}(q) - D_* \underline{\underline{G}})\right]}_{\underline{\underline{L}}\,\underline{c} = 0} \underline{c} = 0$$

Need to determine a N_x+1 by N_x matrix $\underline{\underline{A}}$ that is a function of the known fluxes q, so that $\boxed{\underline{\underline{A}}(q)\, \underline{c} \sim qc}$

Main problem is that c is given at cell centers and q at faces

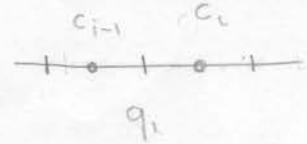
$\Rightarrow$ need approximation of c on interfaces

Advective flux across ith interface



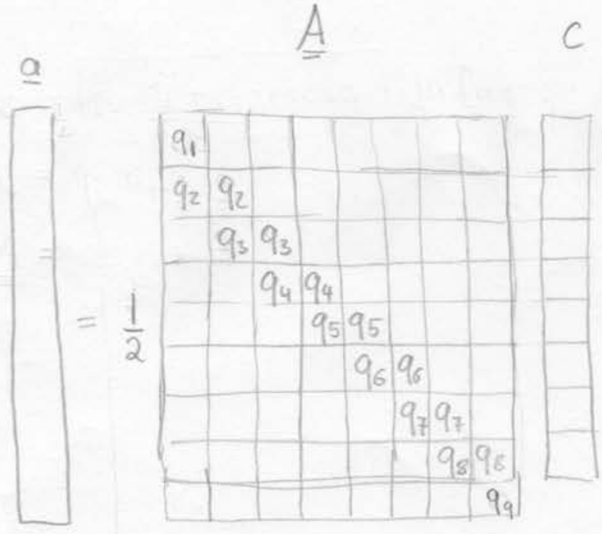
$\boxed{a_i = q_i\, c_{i-\frac{1}{2}}}$ how do we approx $c_{\frac{1}{2}}$

Central flux



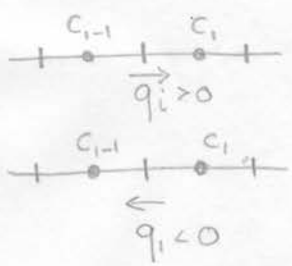
$c_{i\,\frac{1}{2}} = \frac{1}{2}(c_i + c_{i-1})$

$a_i = \frac{1}{2} q_i (c_i + c_{i-1})$

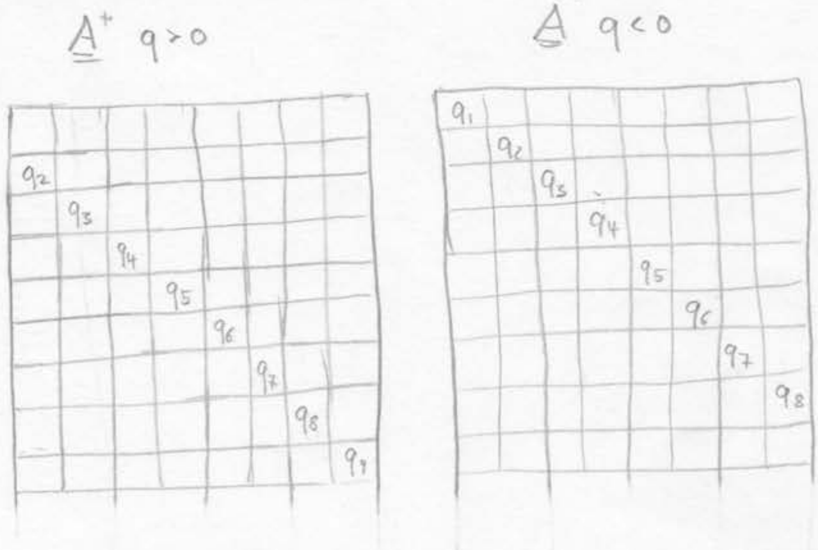


Stability limit: $\Delta x < \frac{2}{Pe} = \frac{2D_H}{qL}$

Upwind flux



$c_{i-\frac{1}{2}} = \begin{cases} c & q_i > 0 \\ c_i & q_i < 0 \end{cases}$



→ need to switch between $\underline{\underline{A}}^+$ & $\underline{\underline{A}}^-$ on a cel by cell basis depending on sign of q

Buid posi ive & negative fux vectors

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qn = min(q(1 Nx), 0),
qp = max(q(2:Nx+1), 0),
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