Advection Eq

3/5 Previous lecture:

- flux reconstruction
  - residual $r = \underline{\underline{L}}\,\underline{u} - f_s$
  - implementation independent of eq
- variable coefficients ϕ_D how do we evolve the porosity?
  - recall mod Helmholtz eq: $\nabla \cdot \underbrace{\phi^n \nabla h}_{\text{faces}} + \underbrace{\phi^m h_D}_{\text{cell}} = \phi^m z_D$ ← ϕ on the diag of matrix
  - → faces: harmonic avg to estimate ϕ on faces

Today:

Start thinking abt porosity evolution

Adv eq: time dependent

first order eq

analytical soln to advection eq

upwind flux

Porosity Evolution

$$\frac{\partial \phi_D}{\partial t_D} + Pe\, \nabla (\underline{v}_D + \phi_D) = \underbrace{Da\, \Gamma_D + \overbrace{\phi^m (h_D - z_D)}^{P_D}}_{0}$$

if $\nabla \cdot v_D = 0$: $\nabla \cdot (v_D + \phi_D) = \underline{v}_D \cdot \nabla_D \phi_D + \phi_D \cancel{\nabla_D \cdot v_D}^{\,0}$

standard form

$$\frac{\partial \phi_D}{\partial t_D} + Pe\, v_D \cdot \nabla \phi_D = 0$$

in 1D

$$\frac{\partial \phi_D}{\partial t_D} + \underbrace{Pe\, v_D}_{v,\ \text{const}} \frac{\partial \phi_D}{\partial x_D} = 0$$

(drop D subscript): $\frac{\partial \phi}{\partial t} + v \frac{\partial \phi}{\partial x} = 0$

Solve w/ Method of Characteristics

PDE: $\frac{\partial \phi}{\partial t} + v \frac{\partial \phi}{\partial x} = 0 \qquad x \in \mathbb{R}$

BC. $\phi(x,0) = \phi_0(x) \qquad t_0 = 0$

Idea: Find a characteristic curve/coord ξ, along which the PDE reduces to an ODE

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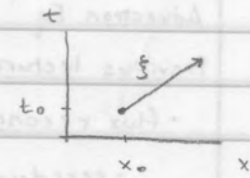
$\phi(x,t) = \phi(x(\xi), t(\xi)) = \Phi(\xi)$



Total change of ϕ along the characteristic

$$\frac{d\Phi}{d\xi} = \frac{\partial\phi}{\partial t}\underbrace{\frac{\partial t}{\partial \xi}}_{1} + \frac{\partial\phi}{\partial x}\underbrace{\frac{\partial x}{\partial \xi}}_{v}$$

Comparison of $\partial\Phi/\partial\xi$ with PDE

$$\frac{\partial\Phi}{\partial\xi} = 0 \qquad \frac{\partial t}{\partial\xi} = 1 \qquad \frac{\partial x}{\partial\xi} = v$$

" characteristic eq "

dividend: $\frac{\partial x}{\partial t} = v$

Whatever the initial ϕ_0 it's carried along the characteristic w/o change

Solve characteristic eq $\quad x - x_0 = v(t - t_0)$

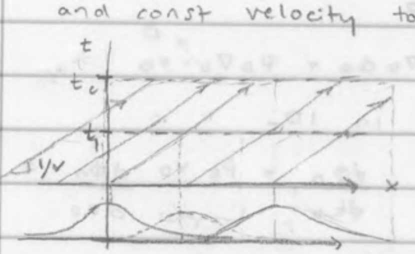
at initial times! $\quad \phi(x = x_0, t = t_0) = \phi_0(x_0)$

substitute char. eq. $\quad x_0 = x - v(t - t_0)$

$$\boxed{\phi(x,t) = \phi_0(x - v(t - t_0))}$$ gen. soln to advec. eq.

travelling wave coords. $\quad x - vt$

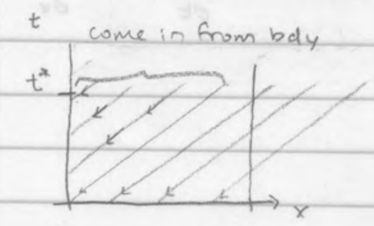
$\Rightarrow$ The initial shape ϕ_0 is shifting / translating with constant shape and const velocity to the right $(v>0)$ or left $(v<0)$



$\frac{dt}{dx} = \frac{1}{v}$ slope of char, $t_0 = 0$

shape doesn't change; translates to right $(v>0)$

Boundary conditions for the advection eq



- don't need BC at outflow $(\underline{v}\cdot\hat{n}_i < 0)$ $(@x=L)$ or if velocity on bdy is 0
- we need BC on inflow side $(\underline{v}\cdot\hat{\underline{n}}_i > 0)$

<u>Note</u> in two-phase flow, in + out flow may depend on changes with phases

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Discretizing the Advection Eq

PDE $\quad \dfrac{d\Phi_D}{dt_D} + Pe\ \nabla\ (\underline{v}_D\, \Phi_D^n) = Da\,\Gamma + \Phi_D^m\,(h_D - z_D)$

for now look at steady problem $\quad \dfrac{d\Phi_D}{dt_D} = 0$

simpilest $\quad Pe\ \nabla_D \cdot \underbrace{(v_D\ \Phi_D)}_{\text{adv. flux}} = Da\,\Gamma$

Discrete form, $\quad Pe\ \underline{\underline{D}} * a_D(\Phi_D) = \underline{f}_s \qquad\qquad \underline{a} = \underline{\underline{A}}\,(\underline{v}_D)\,\Phi_D$

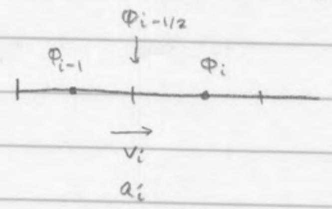
$\underline{a}$ = adv flux vector $\qquad\qquad (Nf_x\ 1) = (Nf_x\ N_x)\ (N_x\ 1)$

$\underline{\underline{A}}(v_D)$ = advection matrix

$$\underbrace{Pe\ \underline{\underline{D}} * \underline{\underline{A}}(v_D)}_{\underline{\underline{L}}} * \Phi_D = \underline{f}_s$$

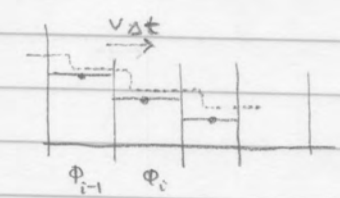
$$\underline{\underline{L}}\ \Phi_D = \underline{f}_s$$

Main purpose of $\underline{\underline{A}}$ is to estimate Φ_D on faces & multiply it by v_D

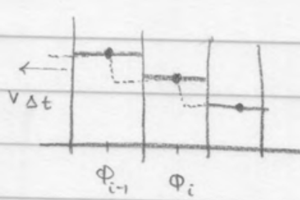


$a_i = v_i\ \Phi_{i-1/2}$

How do we approximate $\Phi_{i-1/2}$?



$v>0$



$v<0$

From analytic soln we know that flux across faces only depends on the Φ_D in direction of the flow that comes from (upwind)

= natural choice is

$$\Phi_{D,i-1/2} \begin{cases} \Phi_{D,i-1} & \text{if } v_D > 0 \\ \Phi_{D,i} & \text{if } v_D < 0 \end{cases}$$